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From finite geometry exact quantities to (elliptic) scattering amplitudes for spin chains: the 1/2-XYZ

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Abstract: Initially, we derive a nonlinear integral equation for the vacuum counting function of the spin 1/2-XYZ chain in the *disordered regime*, thus paralleling similar results by Klümper [1], achieved through a different technique in the *antiferroelectric regime*. In terms of the counting function we obtain the usual physical quantities, like the energy and the transfer matrix (eigenvalues). Then, we introduce a double scaling limit which appears to describe the sine-Gordon theory on cylindrical geometry, so generalising famous results in the plane by Luther [2] and Johnson et al. [3]. Furthermore, after extending the nonlinear integral equation to excitations, we derive scattering amplitudes involving solitons/antisolitons first, and bound states later. The latter case comes out as manifestly related to the Deformed Virasoro Algebra of Shiraishi et al. [4]. Although this nonlinear integral equations framework was contrived to deal with finite geometries, we prove it to be effective for discovering or rediscovering S-matrices. As a particular example, we prove that this unique model furnishes explicitly two S-matrices, proposed respectively by Zamolodchikov [5] and Lukyanov-Mussardo-Penati [6, 7] as plausible scattering description of unknown integrable field theories.

Keywords: Integrable Field Theories, Bethe Ansatz, Exact S-Matrix.

Contents

1. Introduction

By definition, two dimensional (quantum) integrable models possess as many (possibly infinite) commuting independent (conserved) quantities as the degrees of freedom. In the quantum scenario and especially when the number of degrees of freedom is infinite, integrability does not still guarantee the access to exact information through a simple and standard way. Nevertheless, under propitious circumstances simultaneous eigenvectors and corresponding eigenvalues may be obtained by using various methods, among which the Algebraic Bethe Ansatz is a very powerful and customary technique. One reason of its great success may be tracked in the eigenvector construction, though this reveals itself rather cumbersome (but not available in many other Bethe Ansatz versions). Another

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motive may be its huge range of applicability, in particular within the spin chain world. As well-known, this may be thought of as equivalent to classical statistical lattice models (in two dimensions), which often describe important generalisations of scaling quantum field theories. Moreover, although the vocabulary from spin chains to (2D) field theories has still missing items and conjectures in its pages, it seems to have recently gained a new section on spin chains hamiltonians as mixing matrices in four dimensional (super) Yang-Mills theories ([8 − 10] and the development thereof originated).

From the physical point of view, one of the most important effects involves how the properties of the system — and in particular the eigenvalues of the commuting observables — vary with its spatial dimension, i.e. the number of sites in the spin chain case. Besides the relevance of finite size effects in statistical mechanics and condensed matter (e.g. [11] and references therein), the scale of coupling (and then of energy) also seems to be tuned by the spin chain length in the aforementioned 4D field theories of strong interaction.

As for finite size effects, the Non-Linear Integral Equation (NLIE) description — first introduced in [12] for the conformal vacuum and then derived for an off-critical vacuum in [13] by other ways — turned out to be an efficient tool in order to explore scaling properties of conserved charges. Since [14], still concerning features of the vacuum, and [15], regarding excited states, a number of articles was devoted to the analysis of and through a NLIE and mainly follows the route pioneered by Destri and de Vega [13] (cfr. the hungarian lectures [16] for an overview). In this way (which will be ours too), the NLIE stems directly from the Bethe equations and characterises a quantum state by means of a single integral equation in the complex plane. The NLIE has been widely studied for integrable models described by trigonometric-type Bethe equations: for instance, the 1/2-XXZ spin chain [17], the inhomogeneous 1/2-XXZ and sine-Gordon field theory (ground state in [14], excited states in [15]) and the quantum (m)KdV-sG theory [18]. Here instead we wish to understand better a less studied, but more general set-up: the elliptic one. In particular, we choose the 1/2-XYZ spin chain as prototype of our investigation, since it is the direct generalization of the basic trigonometric models and yet not so much complicated to prevent a detailed and profound analysis. Its hamiltonian may also have serious chances to represent a mixing matrix in some gauge theory.

In addition, we have at least another motivation to study *elliptic theories* and especially elliptic spin chains. Again this comes out partially from 2D integrable field theories. The latter are often studied by starting from a scattering S-matrix, which replaces somehow the lagrangian as definition of the theory on the plane (the most significant example being the sine-Gordon study pioneered by both Zamolodchikov [19]). Very often both S-matrix and lagrangian (or the specific perturbation of a conformal field theory) are well identified and tied together. Therefore, the S-matrix has naturally become *the definition* of a field theory, provided it verifies all the field theory axioms and integrability. Nevertheless, the field theory counterpart of a specific S-matrix is sometimes not clear in the lagrangian language and the assumption of the S-matrix as starting point should be only a useful working hypothesis towards further investigations and identifications. The elliptic case is indeed a good example of this phenomenon with its beginning [5] coeval with the trigonometric relative [19], which is crystal clear and paradigmatic since then. On the contrary, the

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whole elliptic scenario is still controversial and very subtle, although some proposals were recently supported [20]. In this article we want to modify the current perspective on the correspondence problem and prove that the 8-vertex hamiltonian is responsible for *all* the known elliptic S-matrices and gives naturally a unitary explanation of their appearance: this effort is willing to give a global view on a fragmented matter. Of course, we might also interpret this as a step towards a field theory description or as an interesting tool to generate elliptic S-matrices (e.g. by raising up the chain spin). In any case, scattering amplitudes may be naturally obtained from the finite size set-up, even though these are clearly defined when the volume is infinite. Moreover, as first pointed out in [15], their derivation requires some information about the excited states. More specifically, if the connection with the kernel of the NLIE may appear a promising gloss in [15] and a comprehensive elaboration in [21] as regards the sine-Gordon field theory, it can be regarded here as a very profitable, predictive and general tool of investigation and analysis.

Concerning the content and organisation of this article, we address the problem of writing a NLIE satisfied by the counting function of the spin 1/2-XYZ chain in the *disordered regime*, by following the more versatile route of Destri and de Vega [13 − 15] (Section 2 about the vacuum and beginning of section 6 about excited states). As for the vacuum state, Klümper wrote down a similar NLIE for the spin 1/2-XYZ chain in the *antiferroelectric regime* and by means of the technique initiated in [12]. In section 3 exact expressions of the eigenvalues of the transfer matrix (and in particular of the energy) are given as nonlinear functionals of the counting function. Both consist of two terms: a contribution proportional to the size and a finite size correction to it. Section 4 is devoted to the trigonometric limit towards the conformal (massless) XXZ chain, both on the NLIE and on the transfer matrix eigenvalues. Separately (section 5), the (trigonometric, but massive) double scaling limit — which yields the sine-Gordon field theory on a cylinder — is performed and then compared to the results of [18]. In section 6, by considering the first excitations upon the vacuum state, we are able to derive the scattering S-matrix of the soliton/antisoliton sector (in the repulsive regime): this matrix turns out to have an elliptic form and coincides with that proposed by general principles (field theory axioms; integrability: factorisation and Yang-Baxter relation) by A.B. Zamolodchikov some time ago [5]. Furthermore, we compute the scattering factor of the lightest soliton-antisoliton bound state in the attractive regime: on the contrary, this manifestly coincides with the structure function of the Deformed Virasoro Algebra (DVA) by Shiraishi-Kubo-Awata-Odake [4] and hence reformulates in XYZ variables the factor hinted by Lukyanov [6]. Then, we thought of this factor as that describing the scattering of the *fundamental elliptic scalar particle*, and therefore as a suitable candidate to describe an elliptic deformation of the sinh-Gordon theory. And indeed, after another mapping of variables, it coincides with the starting definition adopted more recently by Mussardo and Penati [7], although the cumbersome necessary algebra has made this coincidence likely unnoted (cfr. also [20]). Therefore, we find an unified arrangement for both previously known elliptic S-matrices, though the underlying theory is not properly a field theory: on one side the elliptic Zamolodchikov S-matrix, on the other the DVA or Lukyanov-Mussardo-Penati factor. Eventually, some conclusions and many perspectives come to mind and part of both is outlined in section 7.

2. The lattice theory

The spin 1/2-XYZ model with periodic boundary conditions is a (lattice) spin chain with hamiltonian written in terms of Pauli matrices $\sigma^{x,y,z}$

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N}(J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z)\,. \tag{2.1}$$

Here N is the number of lattice sites and because of the periodicity the site $N+1$ is identified with the site 1. The three (real) coupling constants J_x, J_y and J_z may be reparametrised (up to an overall constant) in terms of elliptic functions (as for the notations on elliptic functions, we refer to [22]). In fact, after introducing the (complex) elliptic nome q ($|q|<1$), we may define the modulus $\mathbf{k}$ (and the complementary modulus $\mathbf{k}' = \sqrt{1-\mathbf{k}^2}$) and the associated complete elliptic integral of the first kind $\mathbf{K}$:

$$\begin{aligned}\mathbf{k} &= 4q^{\frac{1}{2}}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4\,,\\ \mathbf{K} &= \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n-1}}{1+q^{2n}}\right)^2\left(\frac{1-q^{2n}}{1-q^{2n-1}}\right)^2\,.\end{aligned} \tag{2.2}$$

We also introduce the parameter $\mathbf{K}'$ such that the elliptic nome reads

$$q = e^{-\pi\frac{\mathbf{K}'}{\mathbf{K}}}\,. \tag{2.3}$$

The theta-functions of nome q, $\theta_{ab}\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right)$, $a,b=0,1$, may now be defined as

$$\begin{aligned}\theta_{00}\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) &= (-qe^{2i\pi u};q^2)(-qe^{-2i\pi u};q^2)(q^2;q^2) = \theta_3\left(\pi u;i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,,\\ \theta_{01}\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) &= (qe^{2i\pi u};q^2)(qe^{-2i\pi u};q^2)(q^2;q^2) = \theta_4\left(\pi u;i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,,\\ \theta_{10}\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) &= 2q^{\frac{1}{4}}\cos\pi u(-q^2e^{2i\pi u};q^2)(-q^2e^{-2i\pi u};q^2)(q^2;q^2) = \theta_2\left(\pi u;i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,,\\ \theta_{11}\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) &= -2q^{\frac{1}{4}}\sin\pi u(q^2e^{2i\pi u};q^2)(q^2e^{-2i\pi u};q^2)(q^2;q^2) = -\theta_1\left(\pi u;i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,,\end{aligned} \tag{2.4}$$

where we have introduced a shorthand notation for the infinite products,

$$(x;a) = \prod_{s=0}^{\infty}(1-xa^s)\,, \tag{2.5}$$

and at the furthest right the expressions in terms of the usual Jacobi theta-functions θ_i, $i=1,2,3,4$. With the exception of Section 6, we will find more convenient to use the alternative functions:

$$H\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) = -\theta_{11}\left(\frac{u}{2\mathbf{K}};i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,,\qquad H_1\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) = \theta_{10}\left(\frac{u}{2\mathbf{K}};i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,, \tag{2.6}$$

$$\Theta\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) = \theta_{01}\left(\frac{u}{2\mathbf{K}};i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,,\qquad \Theta_1\left(u;i\frac{\mathbf{K}'}{\mathbf{K}}\right) = \theta_{00}\left(\frac{u}{2\mathbf{K}};i\frac{\mathbf{K}'}{\mathbf{K}}\right)\,. \tag{2.7}$$

The jacobian elliptic functions are

$$\mathrm{sn}\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right) = \frac{1}{\sqrt{\mathbf{k}}}\frac{H\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right)}{\Theta\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right)}, \quad \mathrm{cn}\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right) = \sqrt{\frac{\mathbf{k}'}{\mathbf{k}}}\frac{H_1\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right)}{\Theta\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right)}, \tag{2.8}$$

$$\mathrm{dn}\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right) = \sqrt{\mathbf{k}'}\frac{\Theta_1\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right)}{\Theta\left(u; i\frac{\mathbf{K}'}{\mathbf{K}}\right)}. \tag{2.9}$$

In order to simplify notations, from now on we will omit the dependence on the elliptic nome, when the elliptic functions have elliptic nome q (2.3).

In terms of jacobian elliptic functions the coupling constants J_x, J_y and J_z in (2.1) are parametrised as

$$J_x = 1 + \mathbf{k}\,\mathrm{sn}^2 2\eta\,, \quad J_y = 1 - \mathbf{k}\,\mathrm{sn}^2 2\eta\,, \quad J_z = \mathrm{cn}2\eta\,\mathrm{dn}2\eta\,, \tag{2.10}$$

where η needs to be real.

The eigenvalues of the spin 1/2-XYZ transfer matrix were firstly found by Baxter [23] by using his auxiliary matrix (Q-operator) technique. Afterwards, in the paper [24] Takhtadjan and Faddeev wrote down also the eigenvectors (at least formally), making use of the Algebraic Bethe Ansatz. In the present paper, we want to restrict our analysis within the disordered regime, i.e.

$$\begin{aligned} &0 < q < 1\,, \\ &0 < \eta < \mathbf{K}\,, \end{aligned} \tag{2.11}$$

which in particular entail $\mathbf{K} > 0$ and $\mathbf{K}' > 0$. With this choice of parameters, we differ from [1], where a NLIE was written (by means of a method far from ours, though) in the antiferroelectric regime: $0 < q < 1$ and $0 < i\eta < \mathbf{K}'$. The Bethe equations obtained in [23, 24] are:

$$\left[\frac{H(i\alpha_j + \eta)\Theta(i\alpha_j + \eta)}{H(i\alpha_j - \eta)\Theta(i\alpha_j - \eta)}\right]^N = -e^{-4\pi i\nu\frac{\eta}{2\mathbf{K}}}\prod_{k=1}^{n}\frac{H(i\alpha_j - i\alpha_k + 2\eta)\Theta(i\alpha_j - i\alpha_k + 2\eta)}{H(i\alpha_j - i\alpha_k - 2\eta)\Theta(i\alpha_j - i\alpha_k - 2\eta)}\,, \tag{2.12}$$

for all $j = 1, \ldots, n$, where ν is an integer. Equations (2.12) are valid when

$$m_1\eta = 2m_2\mathbf{K}\,, \tag{2.13}$$

where m_1 and m_2 are integers, and when $2n = N$ (mod m_1). The corresponding eigenvalues of the transfer matrix are

$$\begin{aligned} \Lambda_N(\alpha) = {} & e^{2\pi i\nu\frac{\eta}{2\mathbf{K}}}\Theta(0)^N H(i\alpha + \eta)^N\Theta(i\alpha + \eta)^N\prod_{j=1}^{n}\frac{H(i\alpha_j - i\alpha + 2\eta)\Theta(i\alpha_j - i\alpha + 2\eta)}{H(i\alpha_j - i\alpha)\Theta(i\alpha_j - i\alpha)} + \\ & + e^{-2\pi i\nu\frac{\eta}{2\mathbf{K}}}\Theta(0)^N H(i\alpha - \eta)^N\Theta(i\alpha - \eta)^N\prod_{j=1}^{n}\frac{H(i\alpha - i\alpha_j + 2\eta)\Theta(i\alpha - i\alpha_j + 2\eta)}{H(i\alpha - i\alpha_j)\Theta(i\alpha - i\alpha_j)}\,. \end{aligned} \tag{2.14}$$

In writing (2.12), (2.14) we used notations by [24].

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2.1 Nonlinear integral equation for the vacuum

We now want to study the Bethe state with lowest energy, i.e. the vacuum. From [24, 25] we know that such a state is given by all real roots ($\alpha_j \in \mathbb{R}$) of the Bethe equations enjoying these additional properties

$$\nu = 0\,, \qquad n = \frac{N}{2}\,, \qquad -\frac{\mathbf{K}'}{2} < \alpha_j < \frac{\mathbf{K}'}{2}\,. \tag{2.15}$$

Using symmetry properties of elliptic functions, we rewrite equations (2.12) as

$$\left[\frac{H(i\alpha_j+\eta)\Theta(i\alpha_j+\eta)}{H(-i\alpha_j+\eta)\Theta(-i\alpha_j+\eta)}\right]^N = (-1)^{1+\frac{N}{2}} \prod_{k=1}^{N/2} \frac{H(i\alpha_j - i\alpha_k + 2\eta)\Theta(i\alpha_j - i\alpha_k + 2\eta)}{H(-i\alpha_j + i\alpha_k + 2\eta)\Theta(-i\alpha_j + i\alpha_k + 2\eta)}\,. \tag{2.16}$$

We then define the function

$$\phi(x,\xi) = i \ln \frac{H(\xi - ix)\Theta(\xi - ix)}{H(\xi + ix)\Theta(\xi + ix)}\,, \quad \xi \in \mathbb{R}\,, \tag{2.17}$$

which has branch points occurring at the points $x = x_{r,s}$ of the complex plane such that

$$|\mathrm{Re} x_{r,s}| = r\mathbf{K}'\,, \qquad |\mathrm{Im} x_{r,s}| = \xi - 2s\mathbf{K}\,, \quad r, s \in \mathbb{Z}\,. \tag{2.18}$$

Therefore $\phi(x,\xi)$ is analytic for x in a strip around the real axis defined by the condition

$$|\mathrm{Im} x| < \min\{\xi, |\xi - 2\mathbf{K}|\}\,. \tag{2.19}$$

Having introduced the function ϕ (2.17), we may write the Bethe equations as

$$iN\phi(\alpha_j, \eta) = \ln(-1)^{1+\frac{N}{2}} + i \sum_{k=1}^{N/2} \phi(\alpha_j - \alpha_k, 2\eta)\,. \tag{2.20}$$

We define the counting function

$$Z_N(x) = N\phi(x,\eta) - \sum_{k=1}^{N/2} \phi(x - \alpha_k, 2\eta)\,, \tag{2.21}$$

which is analytic, as a consequence of (2.19), in the region (containing the real axis)

$$\begin{aligned} &|\mathrm{Im} x| < \eta\,, \quad \text{if} \quad 0 < \eta < \frac{2}{3}\mathbf{K}\,, \\ &|\mathrm{Im} x| < 2\mathbf{K} - 2\eta\,, \quad \text{if} \quad \frac{2}{3}\mathbf{K} < \eta < \mathbf{K}\,. \end{aligned} \tag{2.22}$$

In terms of the counting function, the Bethe roots are identified by the condition

$$Z_N(\alpha_j) = \pi\left(2I_j + 1 + \frac{N}{2}\right)\,, \qquad I_j \in \mathbb{Z}\,. \tag{2.23}$$

For the sake of simplicity, we restrict our analysis to the case $N \in 4\mathbb{N}$, so that

$$e^{iZ_N(\alpha_j)} = -1\,. \tag{2.24}$$

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Now, fundamental property of the vacuum roots, α_j, is not to allow any *missing root* (hole) in equation (2.23) and therefore entail a simple sum property about a function $g(x)$ analytic around the real axis

$$2\pi i \sum_{k=1}^{N/2} g(\alpha_k) = -\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx g'(x-i\epsilon) \ln\left[1+e^{iZ_N(x-i\epsilon)}\right] - \int_{\frac{\mathbf{K}'}{2}}^{-\frac{\mathbf{K}'}{2}} dx g'(x+i\epsilon) \ln\left[1+e^{iZ_N(x+i\epsilon)}\right]. \tag{2.25}$$

In (2.25) $\epsilon > 0$ is a parameter small enough to keep the integration within the analyticity domain[1] (prime means, as usual, derivation). In the limit $\epsilon \to 0$ this equation can be rearranged as

$$2\pi \sum_{k=1}^{N/2} g(\alpha_k) = -\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx g'(x) Z_N(x) + 2\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx g'(x) \mathrm{Im} \ln\left[1+e^{iZ_N(x+i0)}\right]. \tag{2.26}$$

Thanks to the analyticity property (2.22), we may therefore rewrite the vacuum counting function (2.21) in a useful form

$$Z_N(x) = N\phi(x,\eta) - \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} \frac{dy}{2\pi} \phi'(x-y,2\eta) Z_N(y) + \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} \frac{dy}{\pi} \phi'(x-y,2\eta) \mathrm{Im} \ln\left[1+e^{iZ_N(y+i0)}\right], \tag{2.27}$$

where $\phi'(x,\eta)$ is the x-derivative. An inspection of the transformation properties of the elliptic functions

$$\begin{aligned} H(u+i\mathbf{K}') &= iq^{-1/4} e^{-\frac{i\pi u}{2\mathbf{K}}} \Theta(u), \\ \Theta(u+i\mathbf{K}') &= iq^{-1/4} e^{-\frac{i\pi u}{2\mathbf{K}}} H(u), \end{aligned}$$

shows that $\phi(x,\xi)$ and, consequently, $Z_N(x)$ are quasiperiodic in their regions of analyticity (2.19) and (2.22) respectively, with a real quasiperiod $\mathbf{K}'$:

$$\phi(x+\mathbf{K}',\xi) - \phi(x,\xi) = 2\pi\left(1-\frac{\xi}{\mathbf{K}}\right) \quad (x \text{ in strip } (2.19)), \tag{2.28}$$

$$Z_N(x+\mathbf{K}') - Z_N(x) = \pi N \quad (x \text{ in strip } (2.22)). \tag{2.29}$$

Obviously, the derivatives $\phi'(x,\xi)$ and $Z_N'(x)$ are periodic (with period $\mathbf{K}'$). A good way to rearrange (2.27) is to introduce the Fourier coefficient

$$\hat{f}(n) = \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx f(x) e^{2i\frac{n\pi x}{\mathbf{K}'}}, \tag{2.30}$$

for a (quasi)periodic function $f(x)$. In terms of the coefficients $\hat{f}(n)$ the (quasi)periodic function $f(x)$ is expressed as

$$f(x) = \frac{1}{\mathbf{K}'} \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{-2i\frac{n\pi x}{\mathbf{K}'}}, \tag{2.31}$$

[1]We are implicitly assuming the reasonable hypothesis that ϵ may also be made small enough not to allow *spurious* solutions of (2.24) within the integration contour.

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within the principal interval $-\mathbf{K}'/2 < x < \mathbf{K}'$ 2. If we consider a periodic function f and (in general) a quasiperiodic function g, it is well known that their periodic convolution,

$$(f \otimes g)(x) = \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dy f(x-y) g(y) , \tag{2.32}$$

has Fourier coefficients given by the product

$$\widehat{(f \otimes g)}(n) = \hat{f}(n)\hat{g}(n) . \tag{2.33}$$

Bearing this property in mind, we introduce the shorter notation

$$L_N(x) = \mathrm{Im} \ln \left[1 + e^{iZ_N(x+i0)} \right] , \tag{2.34}$$

and may easily prove that the Fourier coefficients of all terms in relation (2.27) satisfy the relation

$$\hat{Z}_N(n) = N\hat{\phi}(n,\eta) - \frac{1}{2\pi}\hat{\phi}'(n,2\eta)\hat{Z}_N(n) + \frac{1}{\pi}\hat{\phi}'(n,2\eta)\hat{L}_N(n) . \tag{2.35}$$

And this immediately entails

$$\hat{Z}_N(n) = N \frac{\hat{\phi}(n,\eta)}{1+\frac{1}{2\pi}\hat{\phi}'(n,2\eta)} + 2 \frac{\frac{1}{2\pi}\hat{\phi}'(n,2\eta)}{1+\frac{1}{2\pi}\hat{\phi}'(n,2\eta)} \hat{L}_N(n) . \tag{2.36}$$

Being the functions $Z_N(x)$, $\phi(x)$ and $L_N(x)$ all odd, their Fourier coefficients are vanishing for $n = 0$ (equation (2.36) is trivially satisfied), and hence we may drop out the zero mode in the series expansion. Therefore for the Fourier series we obtain

$$Z_N(x) = NF(x) + 2 \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dy G(x-y) \mathrm{Im} \ln \left[1 + e^{iZ_N(y+i0)} \right] , \tag{2.37}$$

where we have defined the *forcing term*

$$F(x) = \frac{1}{\mathbf{K}'} \sum_{n=-\infty}^{+\infty} \frac{\hat{\phi}(n,\eta)}{1+\frac{1}{2\pi}\hat{\phi}'(n,2\eta)} e^{-2i\frac{n\pi x}{\mathbf{K}'}} , \tag{2.38}$$

and the operator kernel

$$G(x) = \frac{1}{\mathbf{K}'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\frac{1}{2\pi}\hat{\phi}'(n,2\eta)}{1+\frac{1}{2\pi}\hat{\phi}'(n,2\eta)} e^{-2i\frac{n\pi x}{\mathbf{K}'}} , \tag{2.39}$$

where we have decided to remove the zero mode.

Equation (2.37) is the Non-Linear Integral Equation describing the vacuum of the spin 1/2-XYZ chain in the disordered regime (2.11). Even though it has been derived supposing x real, it is valid in the strip (2.22) thanks to analytic continuation, which would fail at some points at the border of the strip (with real parts given by the first of (2.18)). Now, we need to compute explicitly the Fourier series for the functions involved in (2.37).

2.2 Calculation of Fourier coefficients

First, we recall the Fourier coefficient of $\phi'(x, 2\eta)$ from the results of appendix A. Since x is in region (2.22), we use the first of (A.14) or the second of (A.15) to obtain in either case

$$\hat{\phi}'(n, 2\eta) = 2\pi \frac{\sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} . \tag{2.40}$$

Now, we are ready to compute the coefficients

$$\hat{\phi}(n, \eta) = \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \phi(x, \eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} . \tag{2.41}$$

The zero mode $\hat{\phi}(0, \eta) = 0$ is simply given by $\phi(x, \eta) = -\phi(-x, \eta)$, while the others are given upon integrating by parts and exploiting the quasi-periodicity (2.28):

$$\begin{aligned} \hat{\phi}'(n, \eta) &= \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \phi'(x, \eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} \\ &= (-1)^n \left[\phi\left(\frac{\mathbf{K}'}{2}, \eta\right) - \phi\left(-\frac{\mathbf{K}'}{2}, \eta\right)\right] - \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \phi(x, \eta) \frac{2in\pi}{\mathbf{K}'} e^{2i\frac{n\pi x}{\mathbf{K}'}} \\ &= 2\pi \left(1 - \frac{\eta}{\mathbf{K}}\right) \cos n\pi - \frac{2in\pi}{\mathbf{K}'} \hat{\phi}(n, \eta) . \end{aligned} \tag{2.42}$$

Plugging (2.40) into, we are given the required expression

$$\begin{aligned} \hat{\phi}(0, \eta) &= 0 , \\ \hat{\phi}(n, \eta) &= \frac{i\mathbf{K}'}{n} \left\{ \frac{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} - \left(1 - \frac{\eta}{\mathbf{K}}\right) \cos n\pi \right\} , \ n \neq 0 , \end{aligned} \tag{2.43}$$

which entails, with quasi-periodicity at hand,

$$\phi(x, \eta) = i \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{e^{-2i\frac{n\pi x}{\mathbf{K}'}}}{n} \frac{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} + 2\left(1 - \frac{\eta}{\mathbf{K}}\right) \frac{\pi x}{\mathbf{K}'} . \tag{2.44}$$

And we may re-write this series in a more compact form as

$$\phi(x, \eta) = \sum_{n=-\infty}^{+\infty} \frac{\sin \frac{2n\pi x}{\mathbf{K}'}}{n} \frac{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} , \tag{2.45}$$

whose convergency domain is (like in (2.44)) $|\mathrm{Im} x| < \eta$, containing the strip (2.22).

2.3 Forcing and kernel functions

Given the preceding expressions of $\hat{\phi}(n, \eta)$ (2.43) and of $\hat{\phi}'(n, \eta)$ (2.40), we may explicitly mould the forcing term (2.38) into

$$F(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{i}{n} \frac{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'} - \left(1 - \frac{\eta}{\mathbf{K}}\right) \cos n\pi \sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'} + \sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}} e^{-2i\frac{n\pi x}{\mathbf{K}'}} . \tag{2.46}$$

From (2.39) we read off the Fourier coefficient

$$\hat{G}(n) = \frac{\frac{1}{2\pi}\hat{\phi}'(n, 2\eta)}{1 + \frac{1}{2\pi}\hat{\phi}'(n, 2\eta)} , \tag{2.47}$$

and then simplify this by means of (2.40),

$$\hat{G}(n) = \frac{\sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{2 \sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'} \cosh \frac{2n\eta\pi}{\mathbf{K}'}} , \tag{2.48}$$

whence to obtain the final expression for the kernel

$$G(x) = \frac{1}{\mathbf{K}'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{2 \sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'} \cosh \frac{2n\eta\pi}{\mathbf{K}'}} e^{-2i\frac{n\pi x}{\mathbf{K}'}} . \tag{2.49}$$

Remark 1. *In the case $\eta = \mathbf{K}/2$ the kernel function $G(x)$ vanishes. Therefore the solution of the NLIE is trivial:*

$$Z_N(x) = N \sum_{n=-\infty}^{+\infty} \frac{\sin \frac{2n\pi x}{\mathbf{K}'}}{2n \cosh \frac{n\pi \mathbf{K}}{\mathbf{K}'}} = N\, am\left(2x; i\frac{\mathbf{K}}{\mathbf{K}'}\right) = N\, arcsin\, sn\left(2x; i\frac{\mathbf{K}}{\mathbf{K}'}\right) , \tag{2.50}$$

where the elliptic functions have nome $q' = e^{-\pi\frac{\mathbf{K}}{\mathbf{K}'}}$. This is not a surprise as this case reduces to the free-fermion point $\eta = \pi/4$ in the trigonometric ($q \to 0$, cfr. Section 4) limit.

3. Eigenvalues of the transfer matrix

It is the main aim of this section to write down the eigenvalue of the transfer matrix (2.14) on the vacuum state in terms of the solution of the (vacuum) non-linear integral equation (2.37). Although this might seem a priori a complication, it has at least two main advantages. First, instead of solving a big number of transcendental (Bethe) equations, one should solve a NLIE: this makes numerical computations and approximations much simpler. Second, in the following expressions the bulk terms (proportional to the size N of the system) are clearly separated from their finite size corrections. And as it will be clear later on, the finite size corrections and properties can be singled out in the limit $N \to \infty$.

Again, we start from relation (2.26) which expresses the sum on the vacuum Bethe roots of an arbitrary analytic function g. Using the NLIE, (2.26) may be written as follows

$$\sum_{k=1}^{N/2} g(\alpha_k) = -\frac{N}{2\pi} \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx g'(x) F(x) + \\ + \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} \frac{dx}{\pi} g'(x) \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dy [\delta(x-y) - G(x-y)] \mathrm{Im} \ln \left[1 + e^{iZ_N(y+i0)}\right] . \tag{3.1}$$

In terms of the Fourier coefficients this relation reads as well

$$\sum_{k=1}^{N/2} g(\alpha_k) = -\frac{N}{2\pi\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\hat{F}(n)\hat{g}'(-n) + \\ + \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}}\frac{dx}{\pi}\left\{\frac{1}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\hat{g}'(n)[1-\hat{G}(n)]e^{-\frac{2in\pi x}{\mathbf{K}'}}\right\}\mathrm{Im}\ln\left[1+e^{iZ_N(x+i0)}\right] \quad (3.2)$$

Therefore, once the solution of (2.37) is found, the relation (3.2) may be used in order to compute (very often numerically) the eigenvalues of observables on the vacuum. We remark that the first term in (3.2) is apparently proportional to N. Therefore it gives usually the leading order in the $N \to \infty$ limit. In fact, the second term in (3.2) usually provides the subleading corrections in that limit. It is convenient to define the function constructed upon $g(x)$

$$J_g(x) = \frac{1}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\hat{g}'(n)[1-\hat{G}(n)]e^{-\frac{2in\pi x}{\mathbf{K}'}}\,, \quad (3.3)$$

which appears explicitly in (3.2). Here we decide to add to the definition of $G(x)$ its zero mode, since its inclusion does not change the value of the integral in (3.2) and makes at the same time formulæ more compact. In particular, we want to apply the result (3.2) to the vacuum eigenvalues of the transfer matrix (2.14). In this respect, we may introduce the decomposition

$$\begin{aligned}\Lambda_N(\alpha) &= \Theta(0)^N H(i\alpha+\eta)^N\Theta(i\alpha+\eta)^N\prod_{j=1}^{N/2}\frac{H(i\alpha_j-i\alpha+2\eta)\Theta(i\alpha_j-i\alpha+2\eta)}{H(i\alpha_j-i\alpha)\Theta(i\alpha_j-i\alpha)} + \\ &\quad + \Theta(0)^N H(i\alpha-\eta)^N\Theta(i\alpha-\eta)^N\prod_{j=1}^{N/2}\frac{H(i\alpha-i\alpha_j+2\eta)\Theta(i\alpha-i\alpha_j+2\eta)}{H(i\alpha-i\alpha_j)\Theta(i\alpha-i\alpha_j)} \\ &= \Lambda_N^+(\alpha)+\Lambda_N^-(\alpha)\,.\end{aligned} \quad (3.4)$$

Let us first concentrate on $\ln\Lambda_N^+(\alpha)$. We have that

$$\ln\Lambda_N^+(\alpha) = N\ln\Theta(0) + N\ln H(i\alpha+\eta) + N\ln\Theta(i\alpha+\eta) + \sum_{k=1}^{N/2}\gamma_+(\alpha_k,\alpha)\,, \quad (3.5)$$

where

$$\gamma_+(x,\alpha) = \ln\frac{H(ix-i\alpha+2\eta)\Theta(ix-i\alpha+2\eta)}{H(ix-i\alpha)\Theta(ix-i\alpha)}\,. \quad (3.6)$$

Because of the periodicity property

$$\ln\Lambda_N^+(\alpha+2i\mathbf{K}) = \ln\Lambda_N^+(\alpha)\,, \quad (3.7)$$

we can restrict the complex α to the strip

$$-2\eta < \mathrm{Im}\alpha < 2\mathbf{K}-2\eta\,. \quad (3.8)$$

Upon comparing (3.6) with (2.17), we see that

$$\gamma_+(x,\alpha) = -i\phi(x+i\mathbf{K}-i\eta-\alpha,\mathbf{K}-\eta)\,. \tag{3.9}$$

Recalling (A.13) we easily obtain the Fourier coefficients

$$\hat{\gamma'}_+(n,\alpha) = \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\gamma_+(x,\alpha) e^{\frac{2in\pi x}{\mathbf{K}'}}\,, \tag{3.10}$$

in a form depending on the imaginary part of α

$$\begin{aligned} \hat{\gamma'}_+(n,\alpha) &= 2\pi i \frac{\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{\sinh\frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{\frac{2n\pi}{\mathbf{K}'}(i\alpha-\eta)}\,, \quad \text{if } -2\eta<\mathrm{Im}\alpha<0\,, \\ \hat{\gamma'}_+(n,\alpha) &= -2\pi i \frac{\sinh\frac{2n\eta\pi}{\mathbf{K}'}}{\sinh\frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{\frac{2n\pi}{\mathbf{K}'}(\mathbf{K}+i\alpha-\eta)}\,, \quad \text{if } 0<\mathrm{Im}\alpha<2\mathbf{K}-2\eta\,. \end{aligned} \tag{3.11}$$

In this case, the function J_{γ_+} (3.3) equals

$$\begin{aligned} J_{\gamma_+}(x,\alpha) &= \frac{i\pi}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\frac{e^{\frac{2in\pi}{\mathbf{K}'}(\alpha+i\eta-x)}}{\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\,, \quad \text{if } -2\eta<\mathrm{Im}\alpha<0\,, \\ J_{\gamma_+}(x,\alpha) &= -\frac{i\pi}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\frac{\sinh\frac{2n\eta\pi}{\mathbf{K}'}}{\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}} e^{\frac{2in\pi}{\mathbf{K}'}(\alpha+i\eta-i\mathbf{K}-x)}\,, \quad \text{if } 0<\mathrm{Im}\alpha<2\mathbf{K}-2\eta\,. \end{aligned} \tag{3.12}$$

As a consequence, from relation (2.26) and from the expression for $\hat{F}(n)$ (2.46) we are finally given the two cumbersome expressions:

- if $-2\eta<\mathrm{Im}\alpha<0$:

$$\begin{aligned} \ln\Lambda_N^+(\alpha) &= N\ln\Theta(0)+N\ln H(i\alpha+\eta)+N\ln\Theta(i\alpha+\eta)+ \\ &+N\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{e^{-\frac{2n\pi}{\mathbf{K}'}(i\alpha-\eta)}}{n}\left[\frac{\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{2\cosh\frac{2n\eta\pi}{\mathbf{K}'}\sinh\frac{2n\mathbf{K}\pi}{\mathbf{K}'}}-\frac{\left(1-\frac{\eta}{\mathbf{K}}\right)\cos n\pi}{2\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\right]+ \\ &+\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}}dx\frac{i}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\frac{e^{\frac{2in\pi}{\mathbf{K}'}(\alpha+i\eta-x)}}{\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\mathrm{Im}\ln\left[1+e^{iZ_N(x+i0)}\right]; \end{aligned} \tag{3.13}$$

- if $0<\mathrm{Im}\alpha<2\mathbf{K}-2\eta$:

$$\begin{aligned} \ln\Lambda_N^+(\alpha) &= N\ln\Theta(0)+N\ln H(i\alpha+\eta)+N\ln\Theta(i\alpha+\eta)- \\ &-N\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{e^{-\frac{2n\pi}{\mathbf{K}'}(\mathbf{K}+i\alpha-\eta)}}{n}\times \\ &\times\left[\frac{\sinh\frac{2n\eta\pi}{\mathbf{K}'}}{2\cosh\frac{2n\eta\pi}{\mathbf{K}'}\sinh\frac{2n\mathbf{K}\pi}{\mathbf{K}'}}-\frac{\left(1-\frac{\eta}{\mathbf{K}}\right)\cos n\pi\sinh\frac{2n\eta\pi}{\mathbf{K}'}}{2\cosh\frac{2n\eta\pi}{\mathbf{K}'}\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}\right]- \\ &-\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}}dx\frac{i}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty}\frac{\sinh\frac{2n\eta\pi}{\mathbf{K}'}}{\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\times \\ &\times e^{\frac{2in\pi}{\mathbf{K}'}(\alpha+i\eta-i\mathbf{K}-x)}\mathrm{Im}\ln\left[1+e^{iZ_N(x+i0)}\right]. \end{aligned} \tag{3.14}$$

For what concerns $\ln \Lambda_N^-(\alpha)$, we have that

$$\ln \Lambda_N^-(\alpha) = N \ln \Theta(0) + N \ln H(i\alpha - \eta) + N \ln \Theta(i\alpha - \eta) + \sum_{k=1}^{N/2} \gamma_-(\alpha_k, \alpha) \,, \tag{3.15}$$

where

$$\gamma_-(x, \alpha) = \ln \frac{H(i\alpha - ix + 2\eta)\Theta(i\alpha - ix + 2\eta)}{H(i\alpha - ix)\Theta(i\alpha - ix)} \,. \tag{3.16}$$

We notice that $\gamma_-(x,\alpha) = -\gamma_+(x, \alpha - 2i\eta)$. By using this link, the expression for $\ln \Lambda_N^-(\alpha)$ may be easily obtained from (3.13), (3.14).

3.1 The energy

It may be of some interest to write down the vacuum eigenvalue of the hamiltonian (2.1) in a form which separates the term proportional to N to its finite size correction. From [24] we learn that in general we may extract the eigenvalues of the hamiltonian (of the spin 1/2-XYZ chain) from the eigenvalues of the transfer matrix as

$$E_N = -i\mathrm{sn}2\eta \, \frac{d}{d\alpha} \ln \Lambda_N(\alpha) \Big|_{\alpha=-i\eta} - N \frac{\mathrm{cn}2\eta \mathrm{dn}2\eta}{2} - N\mathrm{sn}2\eta \frac{\Theta'(2\eta)}{\Theta(2\eta)} \,, \tag{3.17}$$

which in the notations of last subsection takes on the form

$$E_N = -i\mathrm{sn}2\eta \, \frac{d}{d\alpha} \ln \Lambda_N^+(\alpha) \Big|_{\alpha=-i\eta} - N \frac{\mathrm{cn}2\eta \mathrm{dn}2\eta}{2} - N\mathrm{sn}2\eta \frac{\Theta'(2\eta)}{\Theta(2\eta)} \,. \tag{3.18}$$

In the particular case of the vacuum, result (3.13) yields the wanted expression

$$\begin{aligned} E_N^{(vac)} &= N\mathrm{sn}2\eta \left[\frac{H'(2\eta)}{H(2\eta)} - N \frac{\mathrm{cn}2\eta\mathrm{dn}2\eta}{2} \right] - \\ &- i\mathrm{sn}2\eta \Bigg\{ -N \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{2\pi i}{\mathbf{K}'} \left[\frac{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{2\cosh \frac{2n\eta\pi}{\mathbf{K}'} \sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} - \frac{\left(1 - \frac{\eta}{\mathbf{K}}\right)\cos n\pi}{2\cosh \frac{2n\eta\pi}{\mathbf{K}'}} \right] + \\ &+ \int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{i}{\mathbf{K}'} \sum_{n=-\infty}^{+\infty} \frac{2in\pi}{\mathbf{K}'} \frac{e^{-\frac{2in\pi x}{\mathbf{K}'}}}{\cosh \frac{2n\eta\pi}{\mathbf{K}'}} \mathrm{Im} \ln \left[1 + e^{iZ_N(x+i0)} \right] \Bigg\} \,. \end{aligned} \tag{3.19}$$

As easily follows from this explicit expression, the last term gives the finite size corrections to the first two terms in the limit $N \to \infty$.

4. Trigonometric limit (i.e. XXZ chain)

In order to have a check on the validity of our results and to prepare the ground for the more refined limit of next Section, it is important to explore the trigonometric limit in which the spin 1/2-XYZ chain reduces to the spin 1/2-XXZ chain ($q \to 0$ or $J_x = J_y = 1$ in (2.1)). We expect to reproduce the results of papers [17, 13]. As the trigonometric limit is also expressed by the limit $\mathbf{K}' \to +\infty$ (therefore $\mathbf{K} \to \pi/2$), a Fourier sum can be replaced by an integral, according to the prescription

$$\frac{1}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty} f\left(\frac{n}{\mathbf{K}'}\right) \to \int_{-\infty}^{+\infty} dp f(p)\,. \tag{4.1}$$

In this Section we aim at writing down the trigonometric limit of the vacuum NLIE (2.37) and of the vacuum eigenvalues of the transfer matrix (3.13), (3.14) and of the energy (3.19). With reference to (2.11), the range of variation of η becomes $0 < \eta < \pi/2$. By applying prescription (4.1) to (2.49), we obtain an integral expression for the kernel function $G(x)$

$$\begin{aligned} G(x) \to G_{sG}(x) &= \int_{-\infty}^{+\infty} dp\, e^{-2ip\pi x} \frac{\sinh\left[2p\left(\frac{\pi}{2}-2\eta\right)\pi\right]}{2\sinh\left[2p\left(\frac{\pi}{2}-\eta\right)\pi\right]\cosh 2p\eta\pi} \\ &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, e^{ipx} \frac{\sinh p\left(\frac{\pi}{2}-2\eta\right)}{2\sinh p\left(\frac{\pi}{2}-\eta\right)\, \cosh p\eta}\,. \end{aligned} \tag{4.2}$$

As a check we may notice that the inclusion of the zero mode of G is irrelevant because

$$\frac{1}{\mathbf{K}'}\hat{G}(0) = \frac{1}{2\mathbf{K}'}\frac{\mathbf{K}-2\eta}{\mathbf{K}-\eta} \tag{4.3}$$

is vanishing in that limit. Analogously the trigonometric limit of

$$\phi'(x,2\eta) = \frac{1}{\mathbf{K}'}\sum_{n=-\infty}^{+\infty} e^{-2i\frac{n\pi x}{\mathbf{K}'}}\hat{\phi}'(n,2\eta) \tag{4.4}$$

may be easily computed starting from (2.40):

$$\phi'(x,2\eta) \to \int_{-\infty}^{+\infty} dp\, e^{-ipx} \frac{\sinh p\left(\frac{\pi}{2}-2\eta\right)}{\sinh p\frac{\pi}{2}}\,. \tag{4.5}$$

For what concerns the forcing term $F(x)$, in the trigonometric limit the oscillating term in (2.46) gives no contribution and leaves us with

$$F(x) \to \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dp}{p}\frac{\sin px}{\cosh p\eta} = \frac{1}{i}\ln\tan\left(\frac{i\pi x}{4\eta}+\frac{\pi}{4}\right) = \arctan\sinh\frac{\pi x}{2\eta}\,. \tag{4.6}$$

Therefore, in the trigonometric limit the NLIE (2.37) takes on the form

$$Z_N(x) = N\arctan\sinh\frac{\pi x}{2\eta} + 2\int_{-\infty}^{+\infty} dy G_{sG}(x-y)\mathrm{Im}\ln\left[1+e^{iZ_N(y+i0)}\right], \tag{4.7}$$

which is the Non-Linear Integral Equation for the spin 1/2-XXZ model in the massless anti-ferromagnetic regime (i.e. $J_x = J_y = 1, J_z = \cos 2\eta$). Moreover, we may straightforwardly perform the limit on the vacuum eigenvalue of the transfer matrix, or better on (3.13), (3.14), with the outcome

- if $-2\eta < \mathrm{Im}\alpha < 0$:

$$\begin{aligned} \ln\Lambda_N^+(\alpha) &= N\int_{-\infty}^{+\infty} \frac{dp}{p} e^{-ip\alpha+p\eta} \frac{\sinh p\left(\frac{\pi}{2}-\eta\right)}{2\cosh p\eta \sinh\frac{p\pi}{2}} + \\ &+ \int_{-\infty}^{+\infty} dx \frac{1}{2\eta\,\sinh\frac{\pi}{2\eta}(\alpha-x)}\mathrm{Im}\ln\left[1+e^{iZ_N(x+i0)}\right]; \end{aligned} \tag{4.8}$$

- if $0 < \mathrm{Im}\alpha < \pi - 2\eta$:

$$\begin{aligned} \ln \Lambda_N^+(\alpha) &= -N \int_{-\infty}^{+\infty} \frac{dp}{p} e^{-ip\alpha + p\eta - p\frac{\pi}{2}} \frac{\sinh p\eta}{2\cosh p\eta \sinh \frac{p\pi}{2}} - \\ &- \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{i \sinh p\eta}{\sinh p\left(\frac{\pi}{2} - \eta\right) \cosh p\eta} \times \\ &\times e^{ip\left(\alpha + i\eta - i\frac{\pi}{2} - x\right)} \mathrm{Im} \ln \left[1 + e^{iZ_N(x+i0)}\right] . \end{aligned} \tag{4.9}$$

Eventually, the energy (3.19) takes on the limit expression

$$\begin{aligned} E \rightarrow \frac{1}{2} N \cos 2\eta - \pi N \sin 2\eta \int_{-\infty}^{+\infty} dp \frac{\sinh 2p\left(\frac{\pi}{2} - \eta\right)\pi}{\sinh p\pi^2 \cosh 2p\eta\pi} + \\ + \frac{\pi}{4\eta^2} \sin 2\eta \int_{-\infty}^{+\infty} dx \frac{\sinh \frac{\pi x}{2\eta}}{\cosh^2 \frac{\pi x}{2\eta}} \mathrm{Im} \ln \left[1 + e^{iZ_N(x+i0)}\right] . \end{aligned} \tag{4.10}$$

And indeed these formulæ coincide with those given in [17, 13] as for the spin 1/2-XXZ model.

5. A double scaling limit: cylinder sine-Gordon

In this Section we want to rewrite the NLIE (2.37) in a particular limit such that it will describe the sine-Gordon field theory on a cylinder. This procedure extends the celebrated results of [2] and [3] which concern how a peculiar infinite length [2] limit of the XYZ reproduces sine-Gordon on the plane. We first notice that expression (2.46) for the forcing term may be separated as

$$NF(x) = N \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{i}{2n} \frac{e^{-2i\frac{n\pi x}{\mathbf{K}'}}}{\cosh \frac{2n\eta\pi}{\mathbf{K}'}} - N \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{i}{2n} \frac{\left(1 - \frac{\eta}{\mathbf{K}}\right) \sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}}{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'} \cosh \frac{2n\eta\pi}{\mathbf{K}'}} e^{-2i\frac{n\pi}{\mathbf{K}'}\left(x - \frac{1}{2}\mathbf{K}'\right)} . \tag{5.1}$$

Then, we shift the variables x and y in (2.37) as $x = x' + \Theta$ and $y = y' + \Theta$, assuming Θ x-, y-independent and positive:

$$\begin{aligned} Z_N(x' + \Theta) &= NF(x' + \Theta) + 2 \int_{-\frac{\mathbf{K}'}{2} - \Theta}^{\frac{\mathbf{K}'}{2} - \Theta} dy' G(x' - y') \mathrm{Im} \ln \left[1 + e^{iZ_N(y' + \Theta + i0)}\right] , \\ & -\frac{\mathbf{K}'}{2} - \Theta < x' < \frac{\mathbf{K}'}{2} - \Theta . \end{aligned} \tag{5.2}$$

Now, the double scaling limit is realised by allowing $\mathbf{K}' \rightarrow +\infty$ when $N \rightarrow \infty$ according to

$$\mathbf{K}' = d \ln DN , \qquad d > 0 , \, D > 0 , \tag{5.3}$$

provided that Θ also diverges as

$$\Theta = c \ln CN , \qquad c > 0 , \, c < \frac{d}{2} , \, C > 0 . \tag{5.4}$$

[2] The length is defined as $R = N\Delta$, with the lattice spacing Δ.

The reason why this is a double limit (namely the lattice spacing Δ is going to zero too, whereas the length $R = N\Delta$ remains finite) is encapsulated in the coefficients C and D and will be clear below. Indeed, since the shift (5.2) we had in mind to define a modified counting function too:

$$Z(x') = \lim_{N\to\infty} Z_N(x'+\Theta)\,. \tag{5.5}$$

Of course the range of variation of the new independent variables x' and y' gets more involved

$$-\frac{d}{2}\ln DN - c\ln CN < x', y' < \frac{d}{2}\ln DN - c\ln CN\,. \tag{5.6}$$

Now, let us derive the limiting value of the forcing term $NF(x'+\Theta)$: the sums in (5.1) may be replaced by integrals according to the rule (4.1), i.e.

$$\begin{aligned} NF(x'+\Theta) = {} & \frac{iN}{2}\int_{-\infty}^{+\infty}\frac{dp}{p}\frac{e^{-2ip\pi(x'+c\ln CN)}}{\cosh 2p\eta\pi} - \\ & -\frac{iN}{2}\int_{-\infty}^{+\infty}\frac{dp}{p}\frac{\left(1-\frac{2\eta}{\pi}\right)\sinh p\pi^2\, e^{-2ip\pi(x'+c\ln CN-\frac{d}{2}\ln DN)}}{\sinh\left[2p\pi\left(\frac{\pi}{2}-\eta\right)\right]\cosh 2p\eta\pi}\,. \end{aligned} \tag{5.7}$$

Because of properties (5.3),(5.4), the first (second) integral of this expression can be calculated by closing the integration contour in the lower (upper) p-complex half plane and avoiding with a semicircle the singularity at $p = 0$. In both cases, the leading contribution is given by the pole with the smallest modulus. When $\pi/6 < \eta < \pi/2$, these poles are at $p = \pm i/(4\eta)$ and therefore the total contribution reads

$$N\pi - 2N^{1-\frac{c\pi}{2\eta}}C^{-\frac{c\pi}{2\eta}}e^{-\frac{x'\pi}{2\eta}} + 2\left(1-\frac{2\eta}{\pi}\right)N^{1+\frac{\pi}{2\eta}\left(c-\frac{d}{2}\right)}\tan\frac{\pi^2}{4\eta}\,C^{\frac{\pi c}{2\eta}}D^{-\frac{\pi d}{4\eta}}e^{\frac{x'\pi}{2\eta}}\,. \tag{5.8}$$

The first term comes from the semicircle around $p = 0$. For technical reasons we need to restrict further the domain of η to within $\frac{\pi}{6} < \eta < \frac{\pi}{4}$, so that $\tan\frac{\pi^2}{4\eta} > 0$, and choose

$$c = \frac{2\eta}{\pi}\,, \qquad d = \frac{8\eta}{\pi}\,, \qquad C = \frac{4}{mR}\,, \qquad D^2 = \frac{16}{m^2R^2}\left(1-\frac{2\eta}{\pi}\right)\tan\frac{\pi^2}{4\eta}\,, \tag{5.9}$$

where m is a positive constant with the dimension of a mass and R is the lattice length. This choice entails

$$NF(x'+\Theta) = N\pi + mR\sinh\frac{\pi x'}{2\eta} + o(N^0)\,, \tag{5.10}$$

where $o(z)$ means "order less than z". Of course, the prescription (4.1) produces on the kernel function $G(x'-y')$ (2.49) the result (4.2) as in the trigonometric XXZ limit. Therefore, we obtain that the limiting counting function $Z(x')$ (5.5) has to satisfy the equation

$$Z(x') = mR\sinh\frac{\pi x'}{2\eta} + 2\int_{-\infty}^{+\infty}dy' G_{sG}(x'-y')\,\mathrm{Im}\ln\left[1+e^{iZ(y'+i0)}\right]\,, \tag{5.11}$$

where, since $N \in 4\mathbb{N}$, the constant $N\pi$ has been reabsorbed in a redefinition of $Z(x')$ and eventually the interval (5.6) has become infinite

$$-\infty < x', y' < +\infty\,. \tag{5.12}$$

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With the identification

$$\eta = \frac{\pi}{2}\left(1 - \frac{b^2}{8\pi}\right), \tag{5.13}$$

the non-linear integral equation (5.11) describes the vacuum of the sine-Gordon field theory with coupling constant b^2 and renormalised mass parameter m (i.e. lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m_0^2}{b^2}\cos b\phi$) on a (space-time) cylinder with spatial circumference R.

Remark 2. *The non-linear integral equation (5.11) has been obtained when $\pi/6 < \eta < \pi/4$. However, it can be considered without problems in the whole region $0 < \eta < \pi/2$, defining everywhere by analytical continuation the state which minimises the energy.*

Remark 3. *We remark that in formula (5.1) one could have made the choices $e^{-2i\frac{n\pi}{\mathbf{K}'}\left(x+\frac{1}{2}\mathbf{K}'\right)}$ or $\frac{1}{2}e^{-2i\frac{n\pi}{\mathbf{K}'}(x+\frac{1}{2}\mathbf{K}')} + \frac{1}{2}e^{-2i\frac{n\pi}{\mathbf{K}'}(x-\frac{1}{2}\mathbf{K}')}$ in the exponential of the last term, in order to express the factor $\cos n\pi$ in (2.46). However, the first choice together with the choice $x = x' - \Theta$, $y = y' - \Theta$, leads to the equality*

$$NF(x' - \Theta) = -N\pi + mR\sinh\frac{\pi x'}{2\eta} + o(N^0). \tag{5.14}$$

Instead, the second choice together with the shifts $x = x' \pm \Theta$, $y = y' \pm \Theta$, gives

$$NF(x' \pm \Theta) = \pm\frac{1}{2}N\pi + mR\sinh\frac{\pi x'}{2\eta} + o(N^0). \tag{5.15}$$

Since the constants $-N\pi$, $\pm\frac{1}{2}N\pi$ are inessential, we eventually obtain again equation (5.11).[3]

Remark 4. *With the choices (5.3), (5.4), (5.9), in the limit $N \to \infty$ the parameters J_x, J_y, J_z (2.10) behave as follows*

$$\begin{aligned} \frac{J_z}{J_x} &\to \cos 2\eta\,, \\ \frac{J_x - J_y}{J_x\, 8\sin^2 2\eta} &\simeq \left(\frac{MR}{4N}\right)^{\frac{8\eta}{\pi}} \to 0\,, \end{aligned} \tag{5.16}$$

where we have introduced a rescaled mass parameter

$$M = \frac{m}{\sqrt{\left|\left(1 - \frac{2\eta}{\pi}\right)\tan\frac{\pi^2}{4\eta}\right|}}. \tag{5.17}$$

Clearly, the famous scaling limit to the sine-Gordon field theory on the full spatial line [2, 3] is now gained from here by sending $R \to \infty$.

Remark 5. *The limit discussed in this section can be applied to the eigenvalues (3.13), (3.14) of the transfer matrix. The calculations of the limits of the Z-independent terms are less straightforward, but they are carried out following a procedure analogous to that used on the forcing term (remembering that due to the redefinition of Z the forcing term $F(x)$ has to be replaced by $F(x) - \pi$). Here we just give the result:*

[3] To be precise this limit was devised in the antiferroelectric regime (where η is purely imaginary).

- *if* $-2\eta < \mathrm{Im}\alpha < 0$*:*

$$\ln \Lambda^+(\alpha) = -mR cotan \frac{\pi^2}{4\eta} \cosh \frac{\pi\alpha}{2\eta} + \\ + \int_{-\infty}^{+\infty} dx \frac{1}{2\eta \sinh \frac{\pi}{2\eta}(\alpha - x)} \mathrm{Im} \ln \left[1 + e^{iZ(x+i0)}\right] ;$$

- *if* $0 < \mathrm{Im}\alpha < \pi - 2\eta$*:*

$$\ln \Lambda^+(\alpha) = -mR \frac{1}{\sin \frac{\pi^2}{4\eta}} \cosh \left[\frac{\pi}{2\eta}\left(i\frac{\pi}{2} - \alpha\right)\right] - \\ - \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{i \sinh p\eta}{\sinh p\left(\frac{\pi}{2} - \eta\right) \cosh p\eta} e^{ip\left(\alpha + i\eta - i\frac{\pi}{2} - x\right)} \times \\ \times \mathrm{Im} \ln \left[1 + e^{iZ(x+i0)}\right] ,$$

which hold when $\pi/4 < \eta < \pi/2$*. Thanks to a sign change of* α*, these unveil the coincidence with the vacuum eigenvalue of the transfer matrix in the sine-Gordon theory,* $\Lambda_{sG}(\alpha)$*, as computed in the second of [18]*

$$\ln \Lambda^+(\alpha) = \ln \Lambda^+_{sG}(-\alpha) . \tag{5.18}$$

6. Scattering theory

As the spin chain becomes infinitely long, $N \to \infty$ and fixed Δ, the infra-red (IR) hamiltonian eigenstates become on mass-shell states. Besides the statistical mechanics interpretation as thermodynamic limit configurations, these may also be thought of as describing the asymptotic particles in the quantum field theory perspective. And in the same view the excited state version of the nonlinear (elliptic) integral equation (2.37), being merely a quantisation rule, gives exactly the scattering amplitudes of the asymptotic states provided that the convolution integral is neglected (when $N \to \infty$). Moreover, integrability reveals itself useful in the factorisation of the general scattering S-matrix into two particle ones, which eventually encode all the on mass-shell information. Actually, this kind of approach was bashfully commenced in [15] and seriously pursued in [21] for what concerns the sine-Gordon theory.

We shall distinguish two cases: 1) the repulsive regime $0 < \eta < \frac{\mathbf{K}}{2}$ (from Section 5 this means indeed $4\pi < b^2 < 8\pi$ in the sine-Gordon limit); 2) the attractive case $\frac{\mathbf{K}}{2} < \eta < \mathbf{K}$ (which covers the remainder $0 < b^2 < 4\pi$). In the first case we only have soliton and antisoliton excitations; in the second case we also have soliton-antisoliton bound states, the so-called breathers: the difference with sine-Gordon is that everything become *deformed* or *elliptic*. In general, the key ingredient shall be the primitive function of the integral kernel (2.49), suitably normalised in the rapidity variable $\tilde{\theta}$ as

$$\chi(\tilde{\theta}) = \int_0^{\frac{\eta}{\mathbf{K}}\tilde{\theta}} dx 2\pi G(x) = i \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{2 \sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'} \cosh \frac{2n\eta\pi}{\mathbf{K}'}} \frac{e^{-2i\frac{n\pi\eta\tilde{\theta}}{\mathbf{K}\mathbf{K}'}}}{n} , \tag{6.1}$$

which holds within the domain $|\mathrm{Im}\tilde{\theta}| < \min\{2\mathbf{K},\, 2\mathbf{K}^2/\eta - 2\mathbf{K}\}$. In the previous equality, we preferred to use the *renormalised* rapidity $\tilde{\theta}$, which is connected to the *bare* rapidity x used in (2.37) by the relation

$$\tilde{\theta} = \frac{\mathbf{K}}{\eta} x\,. \tag{6.2}$$

In fact, the excited state version of the NLIE (2.37) can be now written in a more compact form in terms of $Z_N(\tilde{\theta})$ [4] by following the pattern of [15]:

$$\begin{aligned} Z_N(\tilde{\theta}) =\; & NF(\tilde{\theta}) + \sum_{k=1}^{N_h} \chi(\tilde{\theta} - h_k) - \sum_{k=1}^{N_c} \chi(\tilde{\theta} - c_k) - \sum_{k=1}^{N_w} \chi_{II}(\tilde{\theta} - w_k) - \sum_{k=1}^{N_{sc}} \chi_{II}(\tilde{\theta} - s_k) + \\ & + 2\int_{-\frac{\mathbf{K}\mathbf{K}'}{2\eta}}^{\frac{\mathbf{K}\mathbf{K}'}{2\eta}} d\tilde{\eta}\, \bar{G}(\tilde{\theta} - \tilde{\eta}) \mathrm{Im} \ln \left[1 + e^{iZ_N(\tilde{\eta}+i0)}\right], \end{aligned} \tag{6.3}$$

for an excitation with N_h (real) holes, N_c close (complex) pairs, N_w wide (complex) pairs and N_{sc} (complex) self-conjugated roots.[5] The reason for the appearance of the function χ_{II}, the second determination of χ, is an important technical detail which will be discussed in what follows. We have also introduced a different normalisation for the kernel: $\bar{G}(\tilde{\theta}) = \frac{\eta}{\mathbf{K}} G\left(\frac{\eta}{\mathbf{K}}\tilde{\theta}\right)$. For characterising the state, additional algebraic equations on $Z_N(\tilde{\theta})$ would also be necessary [15]; but these do not affect directly our scattering treatment and then are left out in this context.

6.1 Repulsive regime: $0 < \eta < \frac{\mathbf{K}}{2}$

In this regime of the XYZ model, the two particle asymptotic states are spanned by four independent vectors, which describe respectively the soliton-soliton, the antisoliton-soliton, the soliton-antisoliton and the antisoliton-antisoliton excitations. Although the two-particle S-matrix is, as said before, the building block, we may consider, in general and without extra difficulties, the multi-particle states as those forming a representation space of the Zamolodchikov-Faddeev algebra. In fact, this is a non-commutative algebra with generators $A(\theta)$ and $\bar{A}(\theta)$ satisfying

$$A(\theta_1)A(\theta_2) = S(\theta_1-\theta_2)A(\theta_2)A(\theta_1) + S_a(\theta_1-\theta_2)\bar{A}(\theta_2)\bar{A}(\theta_1) \tag{6.4}$$
$$A(\theta_1)\bar{A}(\theta_2) = S_t(\theta_1-\theta_2)\bar{A}(\theta_2)A(\theta_1) + S_r(\theta_1-\theta_2)A(\theta_2)\bar{A}(\theta_1) \tag{6.5}$$
$$\bar{A}(\theta_1)A(\theta_2) = S_t(\theta_1-\theta_2)A(\theta_2)\bar{A}(\theta_1) + S_r(\theta_1-\theta_2)\bar{A}(\theta_2)A(\theta_1) \tag{6.6}$$
$$\bar{A}(\theta_1)\bar{A}(\theta_2) = S(\theta_1-\theta_2)\bar{A}(\theta_2)\bar{A}(\theta_1) + S_a(\theta_1-\theta_2)A(\theta_2)A(\theta_1)\,. \tag{6.7}$$

[4] This function is actually defined to be $Z_N(\eta\tilde{\theta}/\mathbf{K})$, with a little abuse of notation; similarly it has been done as for the forcing term $F(\tilde{\theta})$.

[5] Generally, the roots come in complex-conjugated pairs because of the real analyticity of the counting function $Z_N(x)$. A pair of complex conjugate roots is said to be close if the absolute value of their imaginary parts is $< \min\{2\mathbf{K},\, 2\mathbf{K}^2/\eta - 2\mathbf{K}\}$, — i.e. the domain of the function $\chi(\tilde{\theta})$ (6.1) —, wide if it is $> \min\{2\mathbf{K},\, 2\mathbf{K}^2/\eta - 2\mathbf{K}\}$. Nevertheless, one root may be *single* if and only if it is either real or self-conjugated. In other words, they behave exactly like sine-Gordon roots [15], although the fixed imaginary part of one self-conjugated root will be $\mathbf{K}^2/\eta$.

Physically, the generator $A(\theta)$ ($\bar{A}(\theta)$) describes a soliton (antisoliton) excitation on the vacuum with rapidity θ. Of course, this heuristic consideration leads immediately to the mentioned particle representations of this algebra: asymptotic in (out) states are created on the vacuum by products in which the generators are arranged with decreasing (increasing) rapidities. As a particular case, the scattering in the two-particle sector may be algebraically described by the four state basis mentioned at the beginning of this Subsection and the functions $S(\theta_{12})$, $S_a(\theta_{12})$, $S_r(\theta_{12})$, $S_t(\theta_{12})$, where $\theta_{12}=\theta_1-\theta_2$, selected as the only non-vanishing elements of the two-particle S-matrix:

$$
\begin{aligned}
{}_{\rm out}\langle\bar{A}(\theta_1)\bar{A}(\theta_2)|\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in} &={}_{\rm out}\langle A(\theta_1)A(\theta_2)|A(\theta_1)A(\theta_2)\rangle_{\rm in}=S(\theta_{12})\,, &(6.8)\\
{}_{\rm out}\langle\bar{A}(\theta_1)A(\theta_2)|\bar{A}(\theta_1)A(\theta_2)\rangle_{\rm in} &={}_{\rm out}\langle A(\theta_1)\bar{A}(\theta_2)|A(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}=S_t(\theta_{12})\,, &(6.9)\\
{}_{\rm out}\langle\bar{A}(\theta_1)A(\theta_2)|A(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in} &={}_{\rm out}\langle A(\theta_1)\bar{A}(\theta_2)|\bar{A}(\theta_1)A(\theta_2)\rangle_{\rm in}=S_r(\theta_{12})\,, &(6.10)\\
{}_{\rm out}\langle\bar{A}(\theta_1)\bar{A}(\theta_2)|A(\theta_1)A(\theta_2)\rangle_{\rm in} &={}_{\rm out}\langle A(\theta_1)A(\theta_2)|\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}=S_a(\theta_{12})\,. &(6.11)
\end{aligned}
$$

From these considerations, it follows that each of the following four states

$$
\frac{1}{\sqrt{2}}(|A(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}\pm|\bar{A}(\theta_1)A(\theta_2)\rangle_{\rm in})\,,\frac{1}{\sqrt{2}}(|A(\theta_1)A(\theta_2)\rangle_{\rm in}\pm|\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}) \tag{6.12}
$$

is preserved by the scattering, i.e. it is an eigenvector of the two particle S-matrix. The respective eigenvalues are the four amplitudes of the scattering processes

$$
\begin{aligned}
&\frac{1}{2}\cdot_{\rm out}\langle A(\theta_1)\bar{A}(\theta_2)\pm\bar{A}(\theta_1)A(\theta_2)|A(\theta_1)\bar{A}(\theta_2)\pm\bar{A}(\theta_1)A(\theta_2)\rangle_{\rm in}\;=\;S_t(\theta_{12})\pm S_r(\theta_{12})\,,\\
&\frac{1}{2}\cdot_{\rm out}\langle A(\theta_1)A(\theta_2)\pm\bar{A}(\theta_1)\bar{A}(\theta_2)|A(\theta_1)A(\theta_2)\pm\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}\;=\;S(\theta_{12})\pm S_a(\theta_{12})\,. \quad (6.13)
\end{aligned}
$$

As a matter of fact, we have written down the eigenvalues of the two-particle S-matrix because we will read off exactly them from the IR limit ($N\to\infty$) of the excited state NLIE (6.3).

Let us start by considering the first eigenvector

$$
\frac{1}{\sqrt{2}}(|A(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}+|\bar{A}(\theta_1)A(\theta_2)\rangle_{\rm in})\,, \tag{6.14}
$$

which describes the symmetric combination of a soliton and an antisoliton, and by computing its eigenvalue

$$
S_+(\theta)=S_t(\theta)+S_r(\theta)\,, \tag{6.15}
$$

with the shorter definition $\theta=\theta_{12}$. As Bethe Ansatz state it is built up by exciting two holes and a (close) complex pair of roots upon the (Fermi-Dirac) real root sea. Now, this configuration entails in a standard manner [15] an excited state nonlinear integral equation of the form (6.3) with a contribution of four $\chi(\tilde{\theta}-\tilde{\theta}_i)$ ($\tilde{\theta}_i=h_1,h_2,c,\bar{c}$ denotes the two holes or the two complex roots respectively). However, as far as the scattering factor is concerned, these four become three once $\tilde{\theta}$ equals any of the hole rapidities. Decisive simplifications occur as N grows: the imaginary parts of the complex roots approach $\pm\mathbf{K}$ respectively, since the expression (6.1) has exactly poles at $|{\rm Im}\tilde{\theta}|=2\mathbf{K}$, and the real part tends to the hole middle point (as anticipated the convolution integral becomes negligible). Thanks to all these facts, we can identify with θ the difference between the hole positions and write down the scattering amplitude as the exponential of the new contribution

$$S_+(\theta) = \exp\left[-i\chi\left(\frac{\theta}{2} - i\mathbf{K}\right) - i\chi\left(\frac{\theta}{2} + i\mathbf{K}\right)\right]\exp\left[i\chi(\theta)\right]. \tag{6.16}$$

Upon exploiting (6.1), this scattering amplitude may be manipulated into

$$S_+(\theta) = \exp\left[-\sum_{n=1}^{\infty} \frac{2i\sinh\frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}\sin\frac{n\pi\eta\theta}{\mathbf{KK}'}}{n\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}\right]\exp\left[\sum_{n=1}^{\infty}\frac{i\sinh\frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}\sin\frac{2n\pi\eta\theta}{\mathbf{KK}'}}{n\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\right], \tag{6.17}$$

or expressed in a more compact manner thanks to the function θ_{11} (2.4) with nome $\exp(-2\pi\frac{\mathbf{K}-\eta}{\mathbf{K}'})$:

$$S_+(\theta) = -e^{\frac{i\pi\eta\theta}{\mathbf{KK}'}}\frac{\theta_{11}\left(-\frac{\eta\theta}{2\mathbf{KK}'} - i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\theta_{11}\left(-\frac{\eta\theta}{2\mathbf{KK}'} + i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}\exp\left[\sum_{n=1}^{\infty}\frac{i\sinh\frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}\sin\frac{2n\pi\eta\theta}{\mathbf{KK}'}}{n\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\right]. \tag{6.18}$$

Let us pass on to the antisymmetric state in the same soliton-antisoliton sector

$$\frac{1}{\sqrt{2}}(|A(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in} - |\bar{A}(\theta_1)A(\theta_2)\rangle_{\rm in}), \tag{6.19}$$

and find out its eigenvalue

$$S_-(\theta) = S_t(\theta) - S_r(\theta). \tag{6.20}$$

It may be analogously described by a configuration with two holes and a wide pair of complex roots and then implies a four χ contribution to the r.h.s. of the excited equation (6.3). While N is growing the two roots tend to be a single self-conjugate wide root with imaginary part $\mathbf{K}^2/\eta$ and real part again fixed by the hole middle point. In fact, this root lies outside the strip $|\mathrm{Im}\tilde{\theta}|\langle 2\mathbf{K}$ (here $0 < \eta < \mathbf{K}/2$) and therefore we need to extend $\chi(\tilde{\theta})$ (6.1) into the so-called second determination, which is given (up to a constant) by

$$\chi_{II}(\tilde{\theta}) = \chi(\tilde{\theta}) + \chi(\tilde{\theta} - \mathrm{sgn}(\mathrm{Im}\tilde{\theta})2i\mathbf{K}). \tag{6.21}$$

Therefore, (6.1) yields explicitly

$$\chi_{II}(\tilde{\theta}) = \frac{1}{i}\log\left[\frac{\theta_{01}\left(\frac{\eta\tilde{\theta}}{\mathbf{KK}'} - \mathrm{sgn}(\mathrm{Im}\tilde{\theta})i\frac{\eta}{\mathbf{K}'} + i\frac{\mathbf{K}-2\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\theta_{01}\left(\frac{\eta\tilde{\theta}}{\mathbf{KK}'} - \mathrm{sgn}(\mathrm{Im}\tilde{\theta})i\frac{\eta}{\mathbf{K}'} - i\frac{\mathbf{K}-2\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}\exp\left(-2\pi\frac{\mathbf{K}-2\eta}{\mathbf{K}'}\mathrm{sgnIm}\tilde{\theta}\right)\right], \tag{6.22}$$

through the entire domain of second determination

$$2\mathbf{K} < |\mathrm{Im}\tilde{\theta}| < \frac{2\mathbf{K}^2}{\eta} - 2\mathbf{K}, \tag{6.23}$$

which is the range of variation of the imaginary part too: $2\mathbf{K} < \mathbf{K}^2/\eta < 2\mathbf{K}^2/\eta - 2\mathbf{K}$ (as $0 < \eta < \mathbf{K}/2$). Therefore, with the same definition of θ the exponential of the new contribution reads

$$S_-(\theta) = \exp\left[-i\chi_{II}\left(\frac{\theta}{2} - i\frac{\mathbf{K}^2}{\eta}\right)\right]\exp\left[i\chi(\theta)\right], \tag{6.24}$$

or equivalently if we make use of (6.22)

$$S_-(\theta) = -e^{\frac{i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}} \frac{\theta_{01}\left(-\frac{\eta\theta}{2\mathbf{K}\mathbf{K}'} - i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\theta_{01}\left(-\frac{\eta\theta}{2\mathbf{K}\mathbf{K}'} + i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)} \exp\left[\sum_{n=1}^{\infty} \frac{i\sinh\frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}\sin\frac{2n\pi\eta\theta}{\mathbf{K}\mathbf{K}'}}{n\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\right]. \quad (6.25)$$

Now, we want to highlight that we have chosen the undetermined constant factor in the definition of $\chi(\tilde{\theta})$ and $\chi_{II}(\tilde{\theta})$ in such a way that $S_+(\theta = 0) = 1$, $S_-(\theta = 0) = -1$: this is indeed what happens for the limiting amplitudes in sine-Gordon.

At this point, we are left with the calculation of the remaining two eigenvalues, $S_1(\theta) = S(\theta) + S_a(\theta)$, $S_2(\theta) = S(\theta) - S_a(\theta)$, corresponding respectively to the two second eigenvectors of (6.12):

$$\frac{1}{\sqrt{2}}(|A(\theta_1)A(\theta_2)\rangle_{\rm in} \pm |\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}). \quad (6.26)$$

But now the similarity with the sine-Gordon configurations clearly fails. In fact, there [15] the antisymmetric combination is simply realised by a Bethe state with two holes and hence would breed a two $\chi(\tilde{\theta}-\tilde{\theta}_j)$ $(j = 1, 2)$ contribution in the r.h.s. of (6.3). And in sine-Gordon theory it gives the eigenvalue of the symmetric combination as well. In other words, both eigenvalues coincide with the scattering amplitude soliton-soliton into soliton-soliton there, being the soliton-soliton into antisoliton-antisoliton event prevented by topological charge conservation ($S_a = 0$). Yet, in the XYZ chain the situation is richer and this degeneracy removed: in physical language the channel soliton-soliton into antisoliton-antisoliton is here allowed and the topological charge not conserved but modulo 4. As a consequence, we need to go along a different route which shall be connected with the existence of a novelty in this scenario: the real periodicity. And indeed, in the elliptic case, we may also consider configurations with two holes and a (close or wide) complex pair in which the position of the real part of the complex pair undergoes a shift by $\pm\frac{1}{2}\mathbf{K}'$ (in the x coordinate) from the middle point of the holes. In terms of $\theta = \frac{\mathbf{K}}{\eta}x$, the scattering factors deriving from these new configurations are given by (6.18) and (6.25) (respectively for symmetric and antisymmetric state) with θ replaced by $\theta \pm \frac{\mathbf{K}\mathbf{K}'}{\eta}$ (the hole contribution $\exp[i\chi(\theta)]$, which should not change, is indeed invariant under those shifts of θ). These simplify further after using $\theta_{11}(z \pm 1/2; \tau) = \mp\theta_{10}(z; \tau)$ and $\theta_{01}(z \pm 1/2; \tau) = \theta_{00}(z; \tau)$ respectively:

$$S_1(\theta) = e^{\frac{i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}} \frac{\theta_{10}\left(-\frac{\eta\theta}{2\mathbf{K}\mathbf{K}'} - i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\theta_{10}\left(-\frac{\eta\theta}{2\mathbf{K}\mathbf{K}'} + i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)} \exp\left[\sum_{n=1}^{\infty} \frac{i\sinh\frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}\sin\frac{2n\pi\eta\theta}{\mathbf{K}\mathbf{K}'}}{n\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\right],$$

$$S_2(\theta) = e^{\frac{i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}} \frac{\theta_{00}\left(-\frac{\eta\theta}{2\mathbf{K}\mathbf{K}'} - i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\theta_{00}\left(-\frac{\eta\theta}{2\mathbf{K}\mathbf{K}'} + i\frac{\eta}{\mathbf{K}'}; 2i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)} \exp\left[\sum_{n=1}^{\infty} \frac{i\sinh\frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}\sin\frac{2n\pi\eta\theta}{\mathbf{K}\mathbf{K}'}}{n\sinh\frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}\cosh\frac{2n\eta\pi}{\mathbf{K}'}}\right]. \quad (6.27)$$

Up to now, we found the scattering amplitudes in the basis vectors describing two particle states (of solitons and antisolitons). However, in order to find a link with the famous Baxter's eight-vertex R-matrix ([23] and [26]), we need to describe the same scattering process in an equivalent way by using a different basis. In this manner, we will also deduce the Zamolodchikov's S-matrix of [5]. In fact, we may introduce the real doublet of particles,

A_1 and A_2, as

$$A(\theta) = \frac{1}{\sqrt{2}}[A_1(\theta) + iA_2(\theta)]\,, \quad \bar{A}(\theta) = \frac{1}{\sqrt{2}}[A_1(\theta) - iA_2(\theta)]\,. \tag{6.28}$$

In terms of these new generators the Zamolodchikov-Faddeev algebra (6.7) looks as follows

$$A_1(\theta_1)A_1(\theta_2) = \sigma(\theta_1-\theta_2)A_1(\theta_2)A_1(\theta_1) + \sigma_a(\theta_1-\theta_2)A_2(\theta_2)A_2(\theta_1) \tag{6.29}$$

$$A_1(\theta_1)A_2(\theta_2) = \sigma_t(\theta_1-\theta_2)A_2(\theta_2)A_1(\theta_1) + \sigma_r(\theta_1-\theta_2)A_1(\theta_2)A_2(\theta_1) \tag{6.30}$$

$$A_2(\theta_1)A_2(\theta_2) = \sigma(\theta_1-\theta_2)A_2(\theta_2)A_2(\theta_1) + \sigma_a(\theta_1-\theta_2)A_1(\theta_2)A_1(\theta_1) \tag{6.31}$$

$$A_2(\theta_1)A_1(\theta_2) = \sigma_t(\theta_1-\theta_2)A_1(\theta_2)A_2(\theta_1) + \sigma_r(\theta_1-\theta_2)A_2(\theta_2)A_1(\theta_1)\,, \tag{6.32}$$

where the σ-amplitudes are related to the S-amplitudes by means of the relations

$$\begin{aligned} 2\sigma &= S_t + S_r + S + S_a\,, \\ 2\sigma_a &= S_t + S_r - S - S_a\,, \\ 2\sigma_t &= S + S_t - S_a - S_r\,, \\ 2\sigma_r &= S + S_r - S_a - S_t\,. \end{aligned} \tag{6.33}$$

Namely, the eigenvalues are given by

$$\begin{aligned} S_+ &= S_t + S_r = \sigma + \sigma_a\,, \\ S_- &= S_t - S_r = \sigma_t - \sigma_r\,, \\ S_1 &= S + S_a = \sigma - \sigma_a\,, \\ S_2 &= S - S_a = \sigma_t + \sigma_r\,. \end{aligned} \tag{6.34}$$

From the expressions for S_+, S_-, S_1, S_2 (6.18), (6.25), (6.27) and after some lengthy calculation, we obtain the σ-amplitudes,

$$\begin{aligned} \sigma(\theta) = &\frac{(x;P,Q^4)^2(Q^2x^{-1};P,Q^4)^2}{(x^{-1};P,Q^4)^2(Q^2x;P,Q^4)^2}\,\cdot \\ &\cdot\frac{(x^{-1};P)(x^{-1};Q^4)(Q^2x;Q^4)(x^{-1}P;P^2)(xP;P^2)(Q^2;P^2)(P^2Q^{-2};P^2)}{(x;P)(x;Q^4)(Q^2x^{-1};Q^4)(x^{-1}Q^2;P^2)(P^2Q^{-2}x;P^2)(P;P^2)^2}\,, \end{aligned} \tag{6.35}$$

where, with a *new definition* of x which holds hereafter,[6]

$$x = \exp\left(\frac{2i\eta\pi\theta}{\mathbf{K}\mathbf{K}'}\right), \qquad Q = \exp\left(-\frac{2\eta\pi}{\mathbf{K}'}\right), \qquad P = \exp\left(4\frac{\eta-\mathbf{K}}{\mathbf{K}'}\pi\right), \tag{6.36}$$

and also the remaining ones

$$\begin{aligned} \frac{\sigma_a(\theta)}{\sigma(\theta)} &= \frac{\mathrm{sn}\left(\frac{2\eta\theta K}{\mathbf{K}\mathbf{K}'};4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\mathrm{sn}\left(\frac{4i\eta K}{\mathbf{K}'};4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}\,, \quad \frac{\sigma_r(\theta)}{\sigma(\theta)} = -\frac{\mathrm{sn}\left(\frac{2\eta\theta K}{\mathbf{K}\mathbf{K}'}-\frac{4i\eta K}{\mathbf{K}'};4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}{\mathrm{sn}\left(\frac{4i\eta K}{\mathbf{K}'};4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)}\,, \\ \frac{\sigma_t(\theta)}{\sigma(\theta)} &= k\,\mathrm{sn}\left(\frac{2\eta\theta K}{\mathbf{K}\mathbf{K}'}-\frac{4i\eta K}{\mathbf{K}'};4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right)\mathrm{sn}\left(\frac{2\eta\theta K}{\mathbf{K}\mathbf{K}'};4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right). \end{aligned} \tag{6.37}$$

[6]Because of the heavy technicality, we preferred to keep the symbol mainly used in the current literature at the cost of this notation abuse.

Of course, the jacobian elliptic function sn has nome P now and also K and k are respectively the first kind complete elliptic integral and the modulus corresponding to the same nome P.

We are now ready to identify our new σ-amplitudes with the entries of the eight-vertex R-matrix. Moreover, Baxter's R-matrix was used more recently in order to define the elliptic algebra $\mathcal{A}_{q_e,p_e}(\widehat{sl}(2)_c)$ [27]. More precisely, either the Baxter's R-matrix $R_B(x_e;q_e,p_e)$ (defined by [27, (22)]) and the R-matrix with scaled nome $R_B^*(x_e;q_e,p_e) = R_B(x_e;q_e,p_e^* = p_e q_e^{-2c})$ are involved in the definition of this elliptic algebra. Therefore the relevance of this identification, which goes through the simple definition

$$S_{rep}(\theta) = \begin{pmatrix} \sigma_r(\theta) & 0 & 0 & \sigma_t(\theta) \\ 0 & \sigma_a(\theta) & \sigma(\theta) & 0 \\ 0 & \sigma(\theta) & \sigma_a(\theta) & 0 \\ \sigma_t(\theta) & 0 & 0 & \sigma_r(\theta) \end{pmatrix} . \tag{6.38}$$

As usual we shall parametrise the elliptic affine parameter x_e by an exponential mapping of the physical rapidity

$$x_e^{-2} = x = \exp\left(\frac{2i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}\right) . \tag{6.39}$$

Furthermore, we need to relate the deformation parameters, $q_e = Q$ and $p_e q_e^{-2} = P$, to finalise our link

$$R_B^*(x_e;q_e,p_e)|_{c=1} = S_{\rm rep}(\theta) . \tag{6.40}$$

Regarding the previous relation it is worth saying that the authors of [27] highlighted the impossibility to find the S-matrix of the XYZ chain in the literature, but reported a conjecture due to F. Smirnov according to which it should be given by $-R_B^*(x_e;q_e,p_e)|_{c=1}$. In this respect, we are now in the position to unveil the mapping between the algebra parameters q_e, p_e and the physical variables of the XYZ chain,

$$q_e = \exp\left(-\frac{2\eta\pi}{\mathbf{K}'}\right) , \quad p_e = \exp\left(-\frac{4\pi\mathbf{K}}{\mathbf{K}'}\right) , \tag{6.41}$$

along with the previous relation (6.39) concerning the rapidity. As a check, we can verify that the range of parameters considered in [27],

$$0 < p_e < q_e^4 , \tag{6.42}$$

corresponds indeed to the repulsive regime.

As a second comparison, we want to show up the way of relating our results on XYZ scattering factors to the Z_4-symmetric S-matrix obtained by A.B. Zamolodchikov [5]. The latter was derived as an elliptic solution of the factorization (Yang-Baxter), unitarity and analyticity conditions depending on the rapidity θ_z and the parameters γ and γ'. Once we identify these with the XYZ chain rapidity and parameters, respectively, in this manner

$$\begin{aligned} &\exp\left(\frac{2i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}\right) = \exp\left(\frac{4i\pi\theta_z}{\gamma'}\right) , \quad \exp\left(-\frac{4\eta\pi}{\mathbf{K}'}\right) = \exp\left(-\frac{4\pi^2}{\gamma'}\right) , \\ &\exp\left(4\frac{\eta-\mathbf{K}}{\mathbf{K}'}\pi\right) = \exp\left(-4\pi\frac{\gamma}{\gamma'}\right) , \end{aligned} \tag{6.43}$$

we obtain that our σ, σ_a, σ_t and σ_r coincide with the homonymous quantities in [5].

The last due comparison is with a Takebe's work [25], which apparently do not contain mention to Zamolodchikov's S-matrix. This paper contains several subtleties which make the verification of its rightness quite difficult (and probably did not contribute to its diffusion). Nevertheless, the method is alternative to ours as it describes XYZ chain states in the $N \to \infty$ limit by converting Bethe equations into a linear equation for the "density" of roots. Then, scattering factors for excited states are read off from the the computation of $o(1)$ (next-to-leading) contribution to the phase shift. The latter arises after a complete circulation of the spatial (periodic) direction. If Takebe's parameters are expressed in terms of ours as

$$\lambda_T = \frac{i\alpha}{2\mathbf{K}}, \qquad \tau_T = \frac{i\mathbf{K}'}{2\mathbf{K}}, \qquad \eta_T = \frac{\eta}{2\mathbf{K}}, \tag{6.44}$$

it is rather an easy matter to reformulate our scattering factors in terms of those in [25, (2.3.54)]:

$$S_+(\theta) = \frac{(2.3.54)}{1}, \qquad S_-(\theta) = \frac{(2.3.54)}{3}, \qquad -S_1(\theta) = \frac{(2.3.54)}{2}, \qquad -S_2(\theta) = \frac{(2.3.54)}{4}, \tag{6.45}$$

where (2.3.54) exploits in [25] the new variables $x_T = -\frac{\lambda_T}{\tau_T}$ and $t_T = \frac{i}{\tau_T}$.

Finally, in order to have a check on our results, it is important to analyse what happens to the XYZ repulsive S-matrix in the scaling sine-Gordon limit, i.e. $\mathbf{K}' \to \infty$, which implies $\mathbf{K} \to \pi/2$ (cfr. Section 5). In this context, thanks to the useful trigonometric limit

$$\lim_{\mathbf{K}' \to \infty} \mathrm{sn}\left(\frac{2\eta\theta K}{\mathbf{K}\mathbf{K}'}; 4i\frac{\mathbf{K}-\eta}{\mathbf{K}'}\right) = \frac{\sinh\frac{\eta\theta}{\pi-2\eta}}{\cosh\frac{\eta\theta}{\pi-2\eta}}, \tag{6.46}$$

we may easily obtain the sine-Gordon values

$$\begin{aligned}
S_+(\theta) &= S_t(\theta) + S_r(\theta) \to -\frac{\sinh\frac{\eta(\theta+i\pi)}{\pi-2\eta}}{\sinh\frac{\eta(\theta-i\pi)}{\pi-2\eta}} e^{i\chi_{sG}(\theta)}, \\
S_-(\theta) &= S_t(\theta) - S_r(\theta) \to -\frac{\cosh\frac{\eta(\theta+i\pi)}{\pi-2\eta}}{\cosh\frac{\eta(\theta-i\pi)}{\pi-2\eta}} e^{i\chi_{sG}(\theta)}, \\
S_1(\theta) &= S(\theta) + S_a(\theta) \to e^{i\chi_{sG}(\theta)}, \\
S_2(\theta) &= S(\theta) - S_a(\theta) \to e^{i\chi_{sG}(\theta)},
\end{aligned} \tag{6.47}$$

with the overall factor

$$e^{i\chi_{sG}(\theta)} = \exp\left[i\int_0^\infty \frac{dk}{k} \frac{\sinh k\left(\frac{\pi^2}{4\eta} - \pi\right)}{\sinh k\left(\frac{\pi^2}{4\eta} - \frac{\pi}{2}\right)\cosh\frac{\pi k}{2}} \sin k\theta\right], \tag{6.48}$$

describing the sine-Gordon scattering soliton-soliton→soliton-soliton [19].

6.2 Attractive regime: $\frac{\mathbf{K}}{2} < \eta < \mathbf{K}$

As anticipated, it is really worth to complete the picture by studying the scattering theory in the attractive regime $\mathbf{K}/2 < \eta < \mathbf{K}$: soliton and antisoliton excitations do not enjoy apart existences only, but also bound states. And we fancy to call these states (elliptic) breathers in analogy with the sine-Gordon locution. Although such bound states — whose number depends indeed on the value of η — shall have different scattering factors, for simplicity's sake we restrict our attention to the lightest one. In fact, we may expect that all these scattering processes should be somehow described by the algebraic structure of our simplest case. At any rate, their account would deserve a separate work [28].

Let an asymptotic state of the lightest breather be created with rapidity θ_i, $i = 1, 2$, by a Zamolodchikov-Faddeev generator $B(\theta_i)$ with exchange relation given by a two-particle amplitude $S_B(\theta_1 - \theta_2)$:

$$B(\theta_1)B(\theta_2) = S_B(\theta_1 - \theta_2)B(\theta_2)B(\theta_1) . \tag{6.49}$$

In our set-up $S_B(\theta_1 - \theta_2)$ derives from considering the thermodynamic limit of the NLIE describing the excitation of two lightest (elliptic) breathers. This configuration with two breathers $B(\theta_i)$ with rapidities θ_i corresponds to adding up, to a sea of real roots, two self-conjugate roots with real part θ_i respectively, and imaginary part $\mathbf{K}^2/\eta$ (limiting value). Then, the NLIE (2.37) takes on the form (discarding the convolution term as usual)

$$Z_N(\tilde{\theta}) = NF(\tilde{\theta}) - \chi_{II}\left(\tilde{\theta} - \theta_1 - i\frac{\mathbf{K}^2}{\eta}\right) - \chi_{II}\left(\tilde{\theta} - \theta_2 - i\frac{\mathbf{K}^2}{\eta}\right), \tag{6.50}$$

when $|\mathrm{Im}\tilde{\theta}| < 2\mathbf{K}^2/\eta - 2\mathbf{K}$ is in the first analyticity strip. Nevertheless, the imaginary part of a self-conjugate root $\frac{\mathbf{K}^2}{\eta} > 2\mathbf{K}^2/\eta - 2\mathbf{K}$ turns out to be outside the first analyticity strip and therefore we had to use the second determination of the function $\chi(\theta)$, i.e.

$$\chi_{II}(\theta) = \chi(\theta) - \chi(\theta - 2i(\mathbf{K}^2/\eta - \mathbf{K})\mathrm{sgnIm}\theta) , \tag{6.51}$$

which holds as the imaginary part $|\mathrm{Im}\theta| > 2\mathbf{K}^2/\eta - 2\mathbf{K}$ lies outside the first analyticity strip. Implementing now the definition (6.1), it follows a compact formula for the functions $\chi_{II}(\theta)$ in (6.50)

$$\chi_{II}(\theta) = i \ln\left[\frac{(x^{-1}P^{-1}Q^2; Q^4)(xQ^2; Q^4)(x^{-1}Q^2; Q^4)(xPQ^2; Q^4)}{(x^{-1}; Q^4)(xP; Q^4)(x^{-1}P^{-1}Q^4; Q^4)(xQ^4; Q^4)}\right], \tag{6.52}$$

where P and Q are defined in (6.36) and

$$x = \exp\left(\frac{2i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}\right). \tag{6.53}$$

As second and last step, we compute $Z_N(\tilde{\theta})$ at one of the self-conjugate roots, for instance

$$\tilde{\theta} = \theta_1 + i\frac{\mathbf{K}^2}{\eta} , \tag{6.54}$$

and expect that the terms involving χ_{II} furnish $i \ln S_B(\theta_1 - \theta_2)$. And again the imaginary part of a self-conjugate root (6.54) $\frac{\mathbf{K}^2}{\eta} > 2\mathbf{K}^2/\eta - 2\mathbf{K}$ lies outside the first analyticity strip of $Z_N(\tilde{\theta})$ itself: therefore we need to consider its second determination as well. In other words, we must introduce the second determination of $\chi_{II}(\theta)$, $\chi_{II}(\theta)_{II}$:

$$Z_N\left(\theta_1 + i\frac{\mathbf{K}^2}{\eta}\right)_{II} = NF\left(\theta_1 + i\frac{\mathbf{K}^2}{\eta}\right)_{II} - \chi_{II}(0)_{II} - \chi_{II}(\theta_1 - \theta_2)_{II} \,. \tag{6.55}$$

We can calculate it by reiterating (6.51) with now $\mathrm{Im}\theta > 0$ and simply obtain

$$\chi_{II}(\theta)_{II} = i \ln\left[-x\frac{(x^{-1}P^{-1}Q^2;Q^4)(x^{-1}P;Q^4)(xP^{-1}Q^4;Q^4)(xPQ^2;Q^4)}{(xP^{-1}Q^2;Q^4)(xP;Q^4)(x^{-1}P^{-1}Q^4;Q^4)(x^{-1}PQ^2;Q^4)}\right], \tag{6.56}$$

where we have eventually identified

$$\theta = \theta_1 - \theta_2, \;\; x = \exp\left(\frac{2i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}\right). \tag{6.57}$$

In conclusion, the scattering factor between the lightest elliptic breathers reads in this XYZ notation

$$\begin{aligned} S_B(\theta) &= -\exp\left[i\chi_{II}(0)_{II}\right]\exp\left[i\chi_{II}(\theta)_{II}\right] \\ &= -\frac{1}{x}\frac{(xP^{-1}Q^2;Q^4)(xP;Q^4)(x^{-1}P^{-1}Q^4;Q^4)(x^{-1}PQ^2;Q^4)}{(x^{-1}P^{-1}Q^2;Q^4)(x^{-1}P;Q^4)(xP^{-1}Q^4;Q^4)(xPQ^2;Q^4)}, \end{aligned} \tag{6.58}$$

with the above θ and x. An overall minus sign has been permitted thanks to the definition of χ_{II} up to a constant and in order to reproduce $S_B(0) = -1$ as in the sine-Gordon field theory.

Despite the cumbersome calculation and the different origin, this scattering factor coincides exactly with the structure function of the Deformed Virasoro Algebra $\mathcal{V}ir_{p_v,q_v}$ introduced by Shiraishi et al. [4] . The Deformed Virasoro Algebra (DVA) is an associative algebra generated by the modes T_n of the current $T(z) = \sum_n T_n z^{-n}$, satisfying the relation

$$f\left(\frac{w}{z}\right)T(z)T(w) - f\left(\frac{z}{w}\right)T(w)T(z) = -\frac{(1-q_v)(1-p_v q_v^{-1})}{1-p_v}\left[\delta\left(\frac{p_v w}{z}\right) - \delta\left(\frac{w}{p_v z}\right)\right], \tag{6.59}$$

where p_v and q_v are complex parameters and

$$f(x_v) = \exp\left[\sum_{n=1}^{\infty}\frac{(1-q_v^n)(1-q_v^{-n}p_v^n)}{1+p_v^n}\frac{x_v^n}{n}\right]. \tag{6.60}$$

Equality (6.59) is to be interpreted as an equality between formal power series, but, as shown in appendix B.1, $f(x_v)$ can be analytically continued to the whole complex plane and this allows us to recast relation (6.59) in the braiding form

$$T(z)T(w) = Y\left(\frac{z}{w}\right)T(w)T(z)\,, \tag{6.61}$$

where

$$Y(x_v) = -\frac{1}{x_v}\frac{(x_v q_v^{-1}p_v;p_v^2)(x_v q_v;p_v^2)(x_v^{-1}q_v^{-1}p_v^2;p_v^2)(x_v^{-1}q_v p_v;p_v^2)}{(x_v^{-1}q_v^{-1}p_v;p_v^2)(x_v^{-1}q_v;p_v^2)(x_v q_v^{-1}p_v^2;p_v^2)(x_v q_v p_v;p_v^2)} \tag{6.62}$$

is the structure function of the DVA. While writing this formula, we have borne in mind the convenient infinite product notation

$$(x;a) = \prod_{s=0}^{\infty} (1 - x a^s)\,, \tag{6.63}$$

introduced in (2.5). Eventually, if we suppose the mapping between spin chain variables and algebra parameters

$$x_v = x^{-1} = \exp\left(-\frac{2i\pi\eta\theta}{\mathbf{K}\mathbf{K}'}\right), \quad q_v = Q^2 P = \exp\left(-\frac{4\pi\mathbf{K}}{\mathbf{K}'}\right), \quad p_v = Q^2 = \exp\left(-\frac{4\eta\pi}{\mathbf{K}'}\right), \tag{6.64}$$

we can see immediately that

$$S_B(\theta) = Y(x_v)\,. \tag{6.65}$$

Therefore, we have proved that the Deformed Virasoro Algebra defines a current, $T(z)$, which closes the Zamolodchikov-Faddeev algebra for the fundamental scalar excitation of the XYZ model.

It is possible to rewrite $S_B(\theta)$ in a different form, which is useful if we want to compare this factor to analogous ones proposed in the literature. We follow the idea of Lukyanov [6], write (6.65), (6.62) in terms of theta-functions, perform a modular transformation on the elliptic nome and eventually obtain (details in appendix B.2)

$$S_B(\theta) = \frac{\theta_{11}\left(\frac{i\theta}{4\mathbf{K}} + \frac{\mathbf{K}}{2\eta}; \frac{i\mathbf{K}'}{4\eta}\right)\theta_{10}\left(\frac{i\theta}{4\mathbf{K}} - \frac{\mathbf{K}}{2\eta}; \frac{i\mathbf{K}'}{4\eta}\right)}{\theta_{11}\left(\frac{i\theta}{4\mathbf{K}} - \frac{\mathbf{K}}{2\eta}; \frac{i\mathbf{K}'}{4\eta}\right)\theta_{10}\left(\frac{i\theta}{4\mathbf{K}} + \frac{\mathbf{K}}{2\eta}; \frac{i\mathbf{K}'}{4\eta}\right)}. \tag{6.66}$$

This is a useful form as it allows us to perform easily the limit $\mathbf{K}' \to \infty$ which describes — after shifting (section 5) — the sine-Gordon model. Indeed, we obtain the first breather scattering factor [19]

$$\lim_{\mathbf{K}'\to\infty} S_B(\theta) = \frac{\sinh\theta + i\sin\pi\xi}{\sinh\theta - i\sin\pi\xi}\,, \tag{6.67}$$

with the usual parameter

$$\xi = \frac{\pi}{2\eta} - 1\,. \tag{6.68}$$

Moreover, it follows from (5.13) that ξ carries on the dependence on the coupling constant b of the sine-Gordon lagrangian:

$$\xi = \frac{b^2}{8\pi - b^2}\,. \tag{6.69}$$

It is an easy job, now, to relate (6.66) to the scattering factor $S_{MP}(\beta)$ proposed by Mussardo and Penati in [7]. We start from the form [7, (6)] for $S_{MP}(\beta)$ and after some manipulations (details in appendix B.3), we can conclude that this expression (6) equals our (6.66), namely $S_{MP}(\beta) = S_B(\theta)$, provided these identifications are taken into account (parameters of [7] are on the l.h.s.)

$$T_{MP} = \frac{\pi\mathbf{K}'}{2\eta}\,, \qquad a_{MP} = \frac{\mathbf{K}}{\eta} - 1\,, \qquad \beta_{MP} = -\frac{\pi\theta}{2\mathbf{K}}\,. \tag{6.70}$$

As an obvious but intriguing byproduct we want to underline the coincidence of the Mussardo-Penati's scattering factor and the structure function of the DVA. We must also report the attempt to deduce the existence of breathers as poles of the Zamolodchikov's S-matrix (6.33) [29].

Remark 6. *The parameters x_v, q_v and p_v of the attractive regime (and of the DVA structure function) inherit a simple expression in terms of the parameters x_e, q_e and p_e of the repulsive regime (and of the R-matrix of the elliptic algebra $\mathcal{A}_{q_e,p_e}(\widehat{sl}(2)_c)$):*

$$x_v = x_e^2\,, \qquad p_v = q_e^2\,, \qquad q_v = p_e\,. \tag{6.71}$$

Although in a different landscape, this seems to be exactly the connection between Vir_{p_v,q_v} and $\mathcal{A}_{q_e,p_e}(\widehat{sl}(2)_c)$ proved in [30].[7]

7. Some conclusions, many prospects

Starting from the (elliptic) Bethe Ansatz equations of the spin 1/2-XYZ chain on a circumference (cfr. for instance [24]), we have written a non-linear integral equation describing describing a generic state (either the vacuum or an excited state) of the model in the disordered regime. Maintaining the size finite, we have studied two different limits: the usual trigonometric limit which gives the spin 1/2-XXZ chain and a double scaling limit which turns out to describe the sine-Gordon field theory on a cylinder. The latter furnishes the generalization of the infinite length scaling limit of [2] and [3] in case of finite volume. Moreover, it has suggested us the heuristic idea that sending to infinity the size of the 1/2-XYZ would infer an elliptic scattering theory, which naturally inherits an elliptic deformation of all the sine-Gordon structures. In fact, any elementary excitation on the vacuum in the repulsive regime can be completely characterised, through a NLIE, by new terms which give rise to the corresponding scattering amplitude. Therefore, the elliptic deformation of the soliton/antisoliton sine-Gordon S-matrix derives from the finite size procedure and the re-expression of this matrix as the Baxter elliptic R-matrix proves what is called in [27] the Smirnov's conjecture, a remarkable connection between representation theory of an elliptic algebra and a scattering S-matrix. Very likely the conjecture may also be extended to more complicated elliptic algebras, being its formulation purely algebraic.

Moreover, we have studied the lightest bound state, i.e. the elliptic deformation of the first sine-Gordon breather, and found that its scattering factor coincides with a proposal by Lukyanov [6] and, in a different context, by Mussardo-Penati [7],[8] provided a mapping between each set of coupling constants and scattering rapidities is given. Actually, the most important result of this calculation is that the scattering factor coming from the spin chain shows manifestly its identity with the braiding factor of the Deformed Virasoro Algebra by Shiraishi-Kubo-Awata-Odake [4] and let us suppose that each heavier breather may produce a braiding algebra with a similar structure (possibly like in [32], but without

[7] As a conjecture this liaison is already in [31] and somehow in [32].

[8] The coincidence of these two was apparently unnoticed beforehand.

one of the squares in the structure function). In this respect, we would like to clarify this point in a ongoing publication [28], since we reckon the basic DVA of [4] as responsible somehow for the algebraic structure and the mass quantisation of the other breathers as well. This conjecture relies upon the identification of the first breather as the fundamental scalar particle and upon the vertex operator construction of DVA given in [33]. We would like to conclude this part about the relevance of the DVA by highlighting how its field theory limit in sine-Gordon should be the quantum version of the non-local symmetry geometrically constructed in [34]. But the field theory inheritance really needs an ad hoc analysis in a separate paper, though the scenario is becoming clearer thanks to these spin chain developments. Nevertheless, it is also worth analysing the whole information obtainable on the lattice system through at least two routes: on the one hand, pretty much is now known about the representation theory of the Deformed Virasoro Algebra (cfr. [31] as a review work); on the other hand, the form factors postulates authorised Mussardo and Penati to work out an entire series of form factors of not better identified "fields" [7] (where also some speculations on the nature of the fields is brought forward in the light of conformal fields structure). By now, a more precise reading of these form factors becomes possible and a field theory interpretation plausible.

In conclusion, a field theory description of the spin chain in the thermodynamic limit, also conveying and re-interpreting previous results, shall be pursued by means of fermionisation and renormalisation group techniques and would contribute to the clarity of the landscape. Towards this aim, valuable results from [20] might be efficiently extracted.

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A. General Fourier coefficient of the function (of x) $\frac{d}{dx}\phi(x+i\alpha,\eta)$

In this appendix we want to compute the Fourier coefficient of the function

$$\frac{d}{dx}\phi(x+i\alpha,\eta)\,, \qquad \alpha\in\mathbb{R}\,. \tag{A.1}$$

From the definition (2.17) of ϕ it follows that we have to consider the expression

$$\begin{aligned}\frac{d}{dx}\phi(x+i\alpha,\eta) = {} & i\frac{d}{dx}\ln H(\eta+\alpha-ix) - i\frac{d}{dx}\ln H(\eta-\alpha+ix) + \\ & + i\frac{d}{dx}\ln\Theta(\eta+\alpha-ix) - i\frac{d}{dx}\ln\Theta(\eta-\alpha+ix)\,.\end{aligned} \tag{A.2}$$

Using formula [22, (8.199.1)] we have:

$$
\begin{aligned}
& i\frac{d}{dx}\ln H(\eta-ix)= \\
&=\frac{\pi}{2\mathbf{K}}\left[\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}-2i\sum_{n=1}^{\infty}\frac{e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)}-e^{-\frac{i\pi}{\mathbf{K}}(\eta-ix)}}{e^{\frac{2\pi\mathbf{K}'n}{\mathbf{K}}}-e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)}-e^{-\frac{i\pi}{\mathbf{K}}(\eta-ix)}+e^{-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}}}\right] \\
&=\frac{\pi}{2\mathbf{K}}\left[\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}-2i\sum_{n=1}^{\infty}\frac{e^{-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}}\left(e^{\frac{i\pi\eta+\pi x}{\mathbf{K}}}-e^{-\frac{i\pi\eta+\pi x}{\mathbf{K}}}\right)}{\left(1-e^{\frac{i\pi\eta+\pi x-2\pi\mathbf{K}'n}{\mathbf{K}}}\right)\left(1-e^{-\frac{i\pi\eta+\pi x+2\pi\mathbf{K}'n}{\mathbf{K}}}\right)}\right].
\end{aligned}
\tag{A.3}
$$

Since $-\frac{\mathbf{K}'}{2}<x<\frac{\mathbf{K}'}{2}$, we can express the denominators as power series:

$$
\begin{aligned}
& i\frac{d}{dx}\ln H(\eta-ix)= \\
&=\frac{\pi}{2\mathbf{K}}\Bigg[\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}- \\
&\qquad -2i\sum_{n=1}^{\infty}\sum_{j,l=0}^{\infty}e^{-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}}\left(e^{\frac{i\pi\eta+\pi x}{\mathbf{K}}}-e^{-\frac{i\pi\eta+\pi x}{\mathbf{K}}}\right)e^{j\frac{i\pi\eta+\pi x-2\pi\mathbf{K}'n}{\mathbf{K}}}e^{-l\frac{i\pi\eta+\pi x+2\pi\mathbf{K}'n}{\mathbf{K}}}\Bigg] \\
&=\frac{\pi}{2\mathbf{K}}\Bigg[\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}- \\
&\qquad -2i\sum_{n=1}^{\infty}\sum_{j,l=0}^{\infty}\left(e^{\frac{i\pi\eta+\pi x}{\mathbf{K}}(1+j-l)-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}(1+j+l)}-e^{-\frac{i\pi\eta+\pi x}{\mathbf{K}}(1+l-j)-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}(1+j+l)}\right)\Bigg] \\
&=\frac{\pi}{2\mathbf{K}}\left\{\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}-2i\sum_{n=1}^{\infty}\sum_{S=0}^{\infty}e^{-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}(1+S)}\sum_{D=-S/2}^{S/2}\left[e^{\frac{i\pi\eta+\pi x}{\mathbf{K}}(1+2D)}-e^{-\frac{i\pi\eta+\pi x}{\mathbf{K}}(1-2D)}\right]\right\} \\
&=\frac{\pi}{2\mathbf{K}}\left\{\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}+2i\sum_{n=1}^{\infty}\sum_{S=0}^{\infty}e^{-\frac{2\pi\mathbf{K}'n}{\mathbf{K}}(1+S)}\left[e^{-\frac{i\pi\eta+\pi x}{\mathbf{K}}(S+1)}-e^{\frac{i\pi\eta+\pi x}{\mathbf{K}}(S+1)}\right]\right\} \\
&=\frac{\pi}{2\mathbf{K}}\left\{\cot\frac{\pi(\eta-ix)}{2\mathbf{K}}+2i\sum_{S=1}^{\infty}\frac{e^{-\frac{2\pi\mathbf{K}'}{\mathbf{K}}S}}{1-e^{-\frac{2\pi\mathbf{K}'}{\mathbf{K}}S}}\left[e^{-\frac{i\pi\eta+\pi x}{\mathbf{K}}S}-e^{\frac{i\pi\eta+\pi x}{\mathbf{K}}S}\right]\right\}.
\end{aligned}
$$

Now, we are ready to perform the integration

$$
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}}dx\left[i\frac{d}{dx}\ln H(\eta+\alpha-ix)-i\frac{d}{dx}\ln H(\eta-\alpha+ix)\right]e^{2i\frac{n\pi x}{\mathbf{K}'}}\,. \tag{A.4}
$$

About the terms depending on the cotangents we have that

$$
\begin{aligned}
&\frac{\pi}{2\mathbf{K}}\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}}dx\left[\cot\frac{\pi(\eta+\alpha-ix)}{2\mathbf{K}}+\cot\frac{\pi(\eta-\alpha+ix)}{2\mathbf{K}}\right]e^{2i\frac{n\pi x}{\mathbf{K}'}}= \\
&\quad=\frac{i\pi}{\mathbf{K}}\sum_{S=1}^{\infty}\left(e^{-\frac{i\pi S(\eta+\alpha)}{\mathbf{K}}}-e^{\frac{i\pi S(\eta-\alpha)}{\mathbf{K}}}\right)\frac{(-1)^n e^{-\frac{\pi S\mathbf{K}'}{2\mathbf{K}}}-1}{\frac{2in\pi}{\mathbf{K}'}-\frac{\pi S}{\mathbf{K}}}+ \\
&\quad+\frac{i\pi}{\mathbf{K}}\sum_{S=1}^{\infty}\left(e^{\frac{i\pi S(\eta+\alpha)}{\mathbf{K}}}-e^{-\frac{i\pi S(\eta-\alpha)}{\mathbf{K}}}\right)\frac{(-1)^n e^{-\frac{\pi S\mathbf{K}'}{2\mathbf{K}}}-1}{\frac{2in\pi}{\mathbf{K}'}+\frac{\pi S}{\mathbf{K}}}\,.
\end{aligned}
\tag{A.5}
$$

On the other hand, the remaining terms yield

$$\frac{i\pi}{\mathbf{K}}\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx e^{2i\frac{n\pi x}{\mathbf{K}'}} \sum_{S=1}^{\infty} \frac{e^{-\frac{2\pi\mathbf{K}'}{\mathbf{K}}S}}{1-e^{-\frac{2\pi\mathbf{K}'}{\mathbf{K}}S}} \left[e^{-\frac{i\pi(\eta+\alpha)+\pi x}{\mathbf{K}}S} - e^{\frac{i\pi(\eta+\alpha)+\pi x}{\mathbf{K}}S} + (x,\alpha\to -x,-\alpha)\right] =$$

$$= \frac{i\pi}{\mathbf{K}}\sum_{S=1}^{\infty} \frac{e^{-\frac{2\pi\mathbf{K}'}{\mathbf{K}}S}}{1-e^{-\frac{2\pi\mathbf{K}'}{\mathbf{K}}S}}(-1)^n \left(e^{-\frac{i\pi(\eta+\alpha)S}{\mathbf{K}}} \frac{e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} - \frac{\pi S}{\mathbf{K}}} - e^{\frac{i\pi(\eta+\alpha)S}{\mathbf{K}}} \frac{e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} + \frac{\pi S}{\mathbf{K}}} - \right.$$

$$\left. - e^{-\frac{i\pi(\eta-\alpha)S}{\mathbf{K}}} \frac{e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} + \frac{\pi S}{\mathbf{K}}} + e^{\frac{i\pi(\eta-\alpha)S}{\mathbf{K}}} \frac{e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} - \frac{\pi S}{\mathbf{K}}} \right). \tag{A.6}$$

Summing these two expressions we get the result for (A.4). However, before doing such a sum, it is convenient to compute the contribution coming from the Θ function:

$$\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \left[i\frac{d}{dx}\ln\Theta(\eta+\alpha-ix) - i\frac{d}{dx}\ln\Theta(\eta-\alpha+ix)\right] e^{2i\frac{n\pi x}{\mathbf{K}'}}. \tag{A.7}$$

Using formula 8.199.4 of [22] we get

$$\begin{aligned} i\frac{d}{dx}\ln\Theta(\eta-ix) &= \frac{2\pi}{\mathbf{K}}\sum_{n=1}^{\infty} \frac{\sin\frac{\pi}{\mathbf{K}}(\eta-ix)}{e^{\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}} - e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)} - e^{-\frac{i\pi}{\mathbf{K}}(\eta-ix)} + e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}}} \\ &= \frac{2\pi}{\mathbf{K}}\sum_{n=1}^{\infty} \frac{\sin\frac{\pi}{\mathbf{K}}(\eta-ix)e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}}}{\left[1-e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}}e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)}\right]\left[1-e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}}e^{-\frac{i\pi}{\mathbf{K}}(\eta-ix)}\right]}. \end{aligned} \tag{A.8}$$

Since $-\mathbf{K}'/2 < x < \mathbf{K}'/2$ we can express as power series the denominators:

$$\begin{aligned} & i\frac{d}{dx}\ln\Theta(\eta-ix) = \\ &\quad = \frac{2\pi}{\mathbf{K}}\sin\frac{\pi}{\mathbf{K}}(\eta-ix)\sum_{n=1}^{\infty} e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}} \sum_{j,l=0}^{\infty} e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}l + \frac{i\pi}{\mathbf{K}}(\eta-ix)l} e^{-\frac{\pi\mathbf{K}'(2n-1)}{\mathbf{K}}j - \frac{i\pi}{\mathbf{K}}(\eta-ix)j} \\ &= \frac{2\pi}{\mathbf{K}}\sin\frac{\pi}{\mathbf{K}}(\eta-ix)\sum_{n=1}^{\infty}\sum_{j,l=0}^{\infty} e^{-\frac{\pi\mathbf{K}'}{\mathbf{K}}(2n-1)(j+l+1)} e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)(l-j)} \\ &= \frac{2\pi}{\mathbf{K}}\sin\frac{\pi}{\mathbf{K}}(\eta-ix)\sum_{n=1}^{\infty}\sum_{S=0}^{\infty}\sum_{D=-S/2}^{S/2} e^{-\frac{\pi\mathbf{K}'}{\mathbf{K}}(2n-1)(S+1)} e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)2D} \\ &= \frac{i\pi}{\mathbf{K}}\sum_{n=1}^{\infty}\sum_{S=0}^{\infty} e^{-\frac{\pi\mathbf{K}'}{\mathbf{K}}(2n-1)(S+1)} \left[e^{-\frac{i\pi}{\mathbf{K}}(\eta-ix)(S+1)} - e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)(S+1)}\right] \\ &= \frac{i\pi}{\mathbf{K}}\sum_{S=1}^{\infty} \frac{e^{\frac{\pi\mathbf{K}'}{\mathbf{K}}S}}{e^{2\frac{\pi\mathbf{K}'}{\mathbf{K}}S}-1}\left[e^{-\frac{i\pi}{\mathbf{K}}(\eta-ix)S} - e^{\frac{i\pi}{\mathbf{K}}(\eta-ix)S}\right]. \end{aligned} \tag{A.9}$$

With the help of (A.9) we can perform the integration involved in (A.7):

$$\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \left[i\frac{d}{dx}\ln\Theta(\eta+\alpha-ix) - i\frac{d}{dx}\ln\Theta(\eta-\alpha+ix)\right] e^{2i\frac{n\pi x}{\mathbf{K}'}} =$$

$$= \frac{i\pi}{\mathbf{K}} \sum_{S=1}^{\infty} \frac{1}{e^{\frac{2\pi\mathbf{K}'}{\mathbf{K}}S} - 1} e^{\frac{\pi\mathbf{K}'}{\mathbf{K}}S} (-1)^n \times$$
$$\times \left(e^{-\frac{i\pi(\eta+\alpha)S}{\mathbf{K}}} \frac{e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} - \frac{\pi S}{\mathbf{K}}} - e^{\frac{i\pi(\eta+\alpha)S}{\mathbf{K}}} \frac{e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} + \frac{\pi S}{\mathbf{K}}} - \right.$$
$$\left. - e^{-\frac{i\pi(\eta-\alpha)S}{\mathbf{K}}} \frac{e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} + \frac{\pi S}{\mathbf{K}}} + e^{\frac{i\pi(\eta-\alpha)S}{\mathbf{K}}} \frac{e^{\frac{\pi\mathbf{K}'S}{2\mathbf{K}}} - e^{-\frac{\pi\mathbf{K}'S}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} - \frac{\pi S}{\mathbf{K}}} \right). \tag{A.10}$$

Now, summing (A.6) and (A.10), we get

$$-\frac{i\pi}{\mathbf{K}} \sum_{S=1}^{\infty} \left[\left(e^{-\frac{i\pi S(\eta+\alpha)}{\mathbf{K}}} - e^{\frac{i\pi S(\eta-\alpha)}{\mathbf{K}}} \right) \frac{(-1)^n e^{-\frac{\pi S\mathbf{K}'}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} - \frac{\pi S}{\mathbf{K}}} - \left(e^{\frac{i\pi S(\eta+\alpha)}{\mathbf{K}}} - e^{-\frac{i\pi S(\eta-\alpha)}{\mathbf{K}}} \right) \frac{(-1)^n e^{-\frac{\pi S\mathbf{K}'}{2\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} + \frac{\pi S}{\mathbf{K}}} \right]. \tag{A.11}$$

We notice that this expression exactly cancels the term in (A.5) proportional to $(-1)^n$. Therefore, we get the following result for the Fourier coefficient of $\frac{d}{dx}\phi(x + i\alpha, \eta)$, $\alpha \in \mathbb{R}$:

$$\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x + i\alpha, \eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} = \frac{i\pi}{\mathbf{K}} \sum_{S=1}^{\infty} \left[\frac{e^{\frac{i\pi S(\eta-\alpha)}{\mathbf{K}}} - e^{-\frac{i\pi S(\eta+\alpha)}{\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} - \frac{\pi S}{\mathbf{K}}} + \frac{e^{-\frac{i\pi S(\eta-\alpha)}{\mathbf{K}}} - e^{\frac{i\pi S(\eta+\alpha)}{\mathbf{K}}}}{\frac{2in\pi}{\mathbf{K}'} + \frac{\pi S}{\mathbf{K}}} \right]. \tag{A.12}$$

The sum over S can be performed using formulæ [22, 1.445.1,2]. The final result depends on the range of values of α and η. We remark that we can remove the restriction to α real, since the imaginary part of α can be easily implemented in the final formula, its effect being a phase. Because of the periodicity property $\phi'(x + 2i\mathbf{K}) = \phi'(x)$, we can restrict α to the interval $-\eta < \mathrm{Re}\alpha < 2\mathbf{K} - \eta$. We remember also that $0 < \eta < \mathbf{K}$ (2.11). We get that, for $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x + i\alpha, \eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} &= 2\pi \frac{\sinh \frac{2n(\mathbf{K}-\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{\frac{2n\pi\alpha}{\mathbf{K}'}}, \quad -\eta < \mathrm{Re}\alpha < \eta, \\
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x + i\alpha, \eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} &= -2\pi \frac{\sinh \frac{2n\eta\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{-\frac{2n\pi(\mathbf{K}-\alpha)}{\mathbf{K}'}}, \quad \eta < \mathrm{Re}\alpha < 2\mathbf{K} - \eta.
\end{aligned} \tag{A.13}$$

On the other hand, formula (A.12) holds also if we replace η with 2η. Since now $0 < 2\eta < 2\mathbf{K}$, in order to compute the sums over S we distinguish the following cases:

- $0 < \eta < \mathbf{K}/2$:

$$\begin{aligned}
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x + i\alpha, 2\eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} &= 2\pi \frac{\sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{\frac{2n\pi\alpha}{\mathbf{K}'}}, \\
&\quad -2\eta < \mathrm{Re}\alpha < 2\eta, \\
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x + i\alpha, 2\eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} &= -2\pi \frac{\sinh \frac{4n\eta\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{-\frac{2n\pi(\mathbf{K}-\alpha)}{\mathbf{K}'}}, \\
&\quad 2\eta < \mathrm{Re}\alpha < 2\mathbf{K} - 2\eta;
\end{aligned} \tag{A.14}$$

- $\mathbf{K}/2 < \eta < \mathbf{K}$:

$$\begin{aligned}
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x+i\alpha, 2\eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} &= 2\pi \frac{\sinh \frac{2n(2\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{\frac{2n\pi}{\mathbf{K}'}(\alpha+\mathbf{K})}\,, \\
&\quad -2\eta < \mathrm{Re}\alpha < 2\eta - 2\mathbf{K}\,, \\
\int_{-\frac{\mathbf{K}'}{2}}^{\frac{\mathbf{K}'}{2}} dx \frac{d}{dx}\phi(x+i\alpha, 2\eta) e^{2i\frac{n\pi x}{\mathbf{K}'}} &= 2\pi \frac{\sinh \frac{2n(\mathbf{K}-2\eta)\pi}{\mathbf{K}'}}{\sinh \frac{2n\mathbf{K}\pi}{\mathbf{K}'}} e^{\frac{2n\pi\alpha}{\mathbf{K}'}}\,, \\
&\quad 2\eta - 2\mathbf{K} < \mathrm{Re}\alpha < 2\mathbf{K} - 2\eta\,.
\end{aligned} \tag{A.15}$$

B. DVA as scattering algebra: a unifying outlook

B.1 XYZ and DVA: an equivalence

In the definition of the Deformed Virasoro Algebra (DVA), the main ingredient is the formal expression

$$f(x_v) = \exp\left[\sum_{n=1}^{\infty} \frac{(1-q_v^n)(1-q_v^{-n}p_v^n)}{1+p_v^n}\frac{x_v^n}{n}\right]. \tag{B.1}$$

The series contained in (B.1) is convergent when $|x_v| < 1$ and if, for instance, the parameters q_v, p_v satisfy $|p_v| < 1$, $|q_v| < 1$ and $|p_v q_v^{-1}| < 1$. In such a case we can sum up the series and define a meromorphic function in the complex plane as its analytic continuation. Now, we want to determine such a function.

We observe that, since $|p_v| < 1$, it is possible to write the denominator in (B.1) as a geometric series:

$$f(x_v) = \exp\left[\sum_{n=1}^{\infty}(1-q_v^n)(1-q_v^{-n}p_v^n)\sum_{s=0}^{\infty}(-1)^s p_v^{ns}\frac{x_v^n}{n}\right]. \tag{B.2}$$

Since $|q_v| < 1$ and $|p_v q_v^{-1}| < 1$, we can exchange the series in (B.2) and perform the summation over n in a suitable range of values for x_v by using the formula

$$\sum_{n=1}^{\infty}\frac{z^n}{n} = \log\frac{1}{1-z}\,, \qquad |z| < 1\,. \tag{B.3}$$

So we obtain

$$f(x_v) = \exp\left[\sum_{s=0}^{\infty}(-1)^s\left(\log\frac{1}{1-x_v p_v^s} - \log\frac{1}{1-x_v q_v^{-1} p_v^{1+s}} - \log\frac{1}{1-x_v q_v p_v^s} + \log\frac{1}{1-x_v p_v^{1+s}}\right)\right]. \tag{B.4}$$

This expression can be rearranged by separating the odd and even s contributions and by some simplifications

$$f(x_v) = \frac{1}{1-x_v}\prod_{s=0}^{\infty}\frac{(1-x_v q_v^{-1} p_v^{1+2s})(1-x_v q_v p_v^{2s})}{(1-x_v q_v^{-1} p_v^{2+2s})(1-x_v q_v p_v^{1+2s})}\,. \tag{B.5}$$

Using notation (2.5) for infinite products, namely

$$(x;a)=\prod_{s=0}^{\infty}(1-xa^s)\,, \tag{B.6}$$

we can write down the compact expression

$$f(x_v)=\frac{1}{1-x_v}\frac{(x_vq_v^{-1}p_v;p_v^2)(x_vq_v;p_v^2)}{(x_vq_v^{-1}p_v^2;p_v^2)(x_vq_vp_v;p_v^2)}\,. \tag{B.7}$$

Now, we define a meromorphic function, the so-called structure function of the DVA, as

$$Y(x_v)=\frac{f(x_v)}{f(x_v^{-1})}=-\frac{1}{x_v}\frac{(x_vq_v^{-1}p_v;p_v^2)(x_vq_v;p_v^2)(x_v^{-1}q_v^{-1}p_v^2;p_v^2)(x_v^{-1}q_vp_v;p_v^2)}{(x_v^{-1}q_v^{-1}p_v;p_v^2)(x_v^{-1}q_v;p_v^2)(x_vq_v^{-1}p_v^2;p_v^2)(x_vq_vp_v;p_v^2)}\,. \tag{B.8}$$

This function enters the exchange algebra

$$T(z)T(w)=Y\left(\frac{z}{w}\right)T(w)T(z)\,, \tag{B.9}$$

which is of course equivalent to the DVA.

B.2 Breather factor $S_B(\theta)$ in Lukyanov's form

We want to elaborate the DVA expression (6.65), (6.62) for $S_B(\theta)$ following an idea by Lukyanov ([6]). First, keeping in mind (6.64), we collect the infinite products of (6.62) within theta-functions of nome $p_v=\exp\left(-\frac{4\eta\pi}{\mathbf{K}'}\right)$:

$$S_B(\theta)=\frac{\theta_{11}\left(-\frac{\eta\theta}{\mathbf{K}\mathbf{K}'}+2i\frac{\mathbf{K}}{\mathbf{K}'};\frac{4i\eta}{\mathbf{K}'}\right)\theta_{01}\left(-\frac{\eta\theta}{\mathbf{K}\mathbf{K}'}-2i\frac{\mathbf{K}}{\mathbf{K}'};\frac{4i\eta}{\mathbf{K}'}\right)}{\theta_{11}\left(-\frac{\eta\theta}{\mathbf{K}\mathbf{K}'}-2i\frac{\mathbf{K}}{\mathbf{K}'};\frac{4i\eta}{\mathbf{K}'}\right)\theta_{01}\left(-\frac{\eta\theta}{\mathbf{K}\mathbf{K}'}+2i\frac{\mathbf{K}}{\mathbf{K}'};\frac{4i\eta}{\mathbf{K}'}\right)}. \tag{B.10}$$

Then, we use the modular transformation

$$\frac{\theta_{11}\left(\alpha;i\frac{K'}{K}\right)}{\theta_{01}\left(\alpha;i\frac{K'}{K}\right)}=i\frac{\theta_{11}\left(-i\frac{K}{K'}\alpha;i\frac{K}{K'}\right)}{\theta_{10}\left(-i\frac{K}{K'}\alpha;i\frac{K}{K'}\right)}, \tag{B.11}$$

which connects theta-functions with nome $p=\exp(-\pi K'/K)$ (on the l.h.s.) with theta-functions with nome $p'=\exp(-\pi K/K')$ (on the r.h.s.). This allows us to re-express $S_B(\theta)$ as follows

$$S_B(\theta)=\frac{\theta_{11}\left(\frac{i\theta}{4\mathbf{K}}+\frac{\mathbf{K}}{2\eta};\frac{i\mathbf{K}'}{4\eta}\right)\theta_{10}\left(\frac{i\theta}{4\mathbf{K}}-\frac{\mathbf{K}}{2\eta};\frac{i\mathbf{K}'}{4\eta}\right)}{\theta_{11}\left(\frac{i\theta}{4\mathbf{K}}-\frac{\mathbf{K}}{2\eta};\frac{i\mathbf{K}'}{4\eta}\right)\theta_{10}\left(\frac{i\theta}{4\mathbf{K}}+\frac{\mathbf{K}}{2\eta};\frac{i\mathbf{K}'}{4\eta}\right)}, \tag{B.12}$$

which is (6.66) of the main text. Upon comparing this relation with (14) of [6], we must identify Lukyanov's variables (on the left) and ours in this manner:

$$\beta_L=\frac{\pi\theta}{2\mathbf{K}}\,,\qquad \xi_L=\frac{\mathbf{K}}{\eta}-1\,,\qquad x_L=\exp\left(-\frac{2\eta\pi}{\mathbf{K}'}\right). \tag{B.13}$$

B.3 Breather factor $S_B(\theta)$ in Mussardo-Penati's form

Here we want to show that Mussardo-Penati scattering factor $S_{MP}(\beta)$ actually coincides with Lukyanov's one and therefore with our $S_B(\theta)$. We start from expression (6) of [7] for $S_{MP}(\beta)$ and simply re-formulate it by expressing the jacobian elliptic functions in terms of theta-functions:

$$S_{MP}(\beta) = \frac{\theta_{11}\left(\frac{\beta-i\pi a}{2i\pi};\frac{iT}{\pi}\right)\theta_{10}\left(\frac{\beta+i\pi a}{2i\pi};\frac{iT}{\pi}\right)\theta_{01}\left(\frac{\beta-i\pi a}{2i\pi};\frac{iT}{\pi}\right)\theta_{00}\left(\frac{\beta+i\pi a}{2i\pi};\frac{iT}{\pi}\right)}{\theta_{11}\left(\frac{\beta+i\pi a}{2i\pi};\frac{iT}{\pi}\right)\theta_{10}\left(\frac{\beta-i\pi a}{2i\pi};\frac{iT}{\pi}\right)\theta_{01}\left(\frac{\beta+i\pi a}{2i\pi};\frac{iT}{\pi}\right)\theta_{00}\left(\frac{\beta-i\pi a}{2i\pi};\frac{iT}{\pi}\right)}. \tag{B.14}$$

As stressed the elliptic nome of these theta-functions is e^{-T}. Now, we use the simple identities (deriving from the definitions),

$$\theta_{11}(u;\tau)\theta_{01}(u;\tau) = \theta_{11}\left(u;\frac{\tau}{2}\right)\frac{(e^{2i\pi\tau};e^{2i\pi\tau})^2}{(e^{i\pi\tau};e^{i\pi\tau})}e^{\frac{i\pi\tau}{8}}, \tag{B.15}$$

$$\theta_{10}(u;\tau)\theta_{00}(u;\tau) = \theta_{10}\left(u;\frac{\tau}{2}\right)\frac{(e^{2i\pi\tau};e^{2i\pi\tau})^2}{(e^{i\pi\tau};e^{i\pi\tau})}e^{\frac{i\pi\tau}{8}}, \tag{B.16}$$

in order to express $S_{MP}(\beta)$ in the final form

$$S_{MP}(\beta) = \frac{\theta_{11}\left(\frac{\beta-i\pi a}{2i\pi};\frac{iT}{2\pi}\right)\theta_{10}\left(\frac{\beta+i\pi a}{2i\pi};\frac{iT}{2\pi}\right)}{\theta_{11}\left(\frac{\beta+i\pi a}{2i\pi};\frac{iT}{2\pi}\right)\theta_{10}\left(\frac{\beta-i\pi a}{2i\pi};\frac{iT}{2\pi}\right)}. \tag{B.17}$$

Indeed, we notice that this coincides with $S_B(\theta)$ (6.66), provided we link Mussardo-Penati's variables (on the l.h.s.) with ours (on the r.h.s.) according to

$$T_{MP} = \frac{\pi\mathbf{K}'}{2\eta}, \qquad a_{MP} = \frac{\mathbf{K}}{\eta} - 1, \qquad \beta_{MP} = -\frac{\pi\theta}{2\mathbf{K}}. \tag{B.18}$$

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